

A new obstruction to the extension problem for Sobolev maps between manifolds

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*Dedicated to Haïm Brezis on the occasion of his 70th birthday.
His work and friendship are a permanent
source of inspiration and motivation.*

Abstract

The main result of the present paper, combined with earlier results of Hardt and Lin [10] settles the extension problem for $W^{1,p}(\mathcal{M}, \mathcal{N})$, where \mathcal{M} and \mathcal{N} are compact riemannian manifolds, \mathcal{M} having non-empty smooth boundary and assuming moreover that \mathcal{N} is simply connected. The main question which is studied is the following: Given a map in the trace space $W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$, does it possess an extension in $W^{1,p}(\mathcal{M}, \mathcal{N})$? We show that the answer is negative in the case $\mathfrak{p}_c + 1 \leq p < m = \dim \mathcal{M}$, where the number \mathfrak{p}_c is related to the topology of \mathcal{N} and is defined in (4). We also adress the case \mathcal{N} is not simply connected, providing various results and rising some open questions. In particular, we stress in that case the relationship between the extension problem and the lifting problem to the universal covering manifold.

1 Introduction

{intro}

1.1 The extension problem in the Sobolev class

{mainresult}

We consider in this paper two compact riemannian manifolds \mathcal{M} and \mathcal{N} with \mathcal{N} isometrically embedded in some euclidean space \mathbb{R}^ℓ , \mathcal{M} having a nonempty smooth boundary. For given $1 < p < \infty$, we consider the Sobolev space $W^{1,p}(\mathcal{M}, \mathcal{N})$ of maps between \mathcal{M} and \mathcal{N} defined by

$$W^{1,p}(\mathcal{M}, \mathcal{N}) = \{u \in W^{1,p}(\mathcal{M}, \mathbb{R}^\ell), \ u(x) \in \mathcal{N} \text{ for almost every } x \in \mathcal{M}\}.$$

By the trace theorem, the restriction of any map in $W^{1,p}(\mathcal{M}, \mathcal{N})$ is a map in the trace space $W^{1-1/p,p}(\mathcal{M}, \mathcal{N})$ defined by

$$W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N}) = \{u \in W^{1-1/p,p}(\partial\mathcal{M}, \mathbb{R}^\ell), \ u(x) \in \mathcal{N} \text{ for almost every } x \in \partial\mathcal{M}\}, \quad (1) \quad \{\text{palace}\}$$

where the space $W^{1-1/p,p}(\partial\mathcal{M}, \mathbb{R}^\ell)$ is the standard trace space of maps from $\partial\mathcal{M}$ to \mathbb{R}^ℓ for which the norm $\|\cdot\|_{1-1/p,p}$ is finite. The norm $\|u\|_{1-1/p,p}$ is given by

$$\|u\|_{1-1/p,p} = \|u\|_{L^p(\partial\mathcal{M})} + |u|_{1-1/p,p}$$

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where the semi-norm $|\cdot|_{1-1/p,p}$ writes

$$|u|_{1-1/p,p} = \left(\int_{\partial\mathcal{M}} \int_{\partial\mathcal{M}} \frac{|u(x) - u(y)|^p}{|x - y|^{p+m-2}} dx dy \right)^{\frac{1}{p}}. \quad (2) \quad \{\text{seminorm}\}$$

Given any map g in $W^{1-1/p,p}(\partial\mathcal{M}, \mathbb{R}^\ell)$, it is well-known that there exists an extension u of g to the full domain \mathcal{M} such that $u \in W^{1,p}(\mathcal{M}, \mathbb{R}^\ell)$ and $u = g$ on $\partial\mathcal{M}$ in the sense of the trace operator. In the case we assume furthermore that the values of g are constrained to belong to \mathcal{N} so that the map g belongs to the space $W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$, a natural question, which has already been raised in several places in the litterature, is to determine whether we may find such an extension u satisfying moreover the constraint on the target, that is $u(x) \in \mathcal{N}$ for almost every $x \in \mathcal{M}$. Following the notation introduced in [3] we consider the subset $\mathcal{T}^p(\partial\mathcal{M}, \mathcal{N})$ of $W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$ defined by

$$\mathcal{T}_{\text{ext}}^p(\partial\mathcal{M}, \mathcal{N}) \equiv \{u \in W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N}) \text{ s.t. } \exists U \in W^{1,p}(\mathcal{M}, \mathcal{N}) \text{ such that } U = u \text{ on } \partial\mathcal{M}\}.$$

The extension problem for Sobolev mappings then can be rephrased as:

$$(\mathcal{Q})_p \text{ Under which conditions on } \mathcal{M}, \mathcal{N} \text{ and } p \text{ do we have } \mathcal{T}^p(\partial\mathcal{M}, \mathcal{N}) = W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})?$$

It follows from Sobolev embedding that in the case $p > m = \dim \mathcal{M}$ that maps in $W^{1,p}(\mathcal{M}, \mathcal{N})$ are in fact continuous so that the answer to question \mathcal{Q}_p completely reduces to the corresponding extension problem for continuous maps between \mathcal{M} and \mathcal{N} , a problem in topology which might present significant difficulties, depending on the nature of \mathcal{M} . The same answer holds for the *limiting case* $p = \dim \mathcal{M}$ (see Theorems 1 and 2 in [3]). We therefore restrict ourselves to the case $p < m$. Since the nature of our results is quite different in the two cases, we need to distinguish the case when \mathcal{N} is simply connected from the case \mathcal{N} is not.

1.2 Statement of the result in the case \mathcal{N} is simply connected

We assume here that \mathcal{N} is simply connected, that is

$$\pi_1(\mathcal{N}) = \{0\}. \quad (3) \quad \{\text{simplet}\}$$

It turns out in the case (3) holds, *somewhat surprisingly*, that question $(\mathcal{Q})_p$ has a complete answer which depends only on p and the topological properties of the target manifold \mathcal{N} . In order to state our result, we introduce the integer

$$\mathfrak{p}_c(\mathcal{N}) = \inf\{j \in \mathbb{N}^*, \pi_j(\mathcal{N}) \neq \{0\}\}. \quad (4) \quad \{\text{deftrp}\}$$

For instance if the manifold \mathcal{N} is the n -dimensional sphere \mathbb{S}^n , with $n \geq 2$ so that (3) holds, then $\mathfrak{p}_c(\mathbb{S}^n) = n$. Notice that, since \mathcal{N} is assumed to be compact $\mathfrak{p}_c(\mathcal{N}) < +\infty$, and since it is assumed to be simply connected $1 < \mathfrak{p}_c(\mathcal{N})$. Our main result in the case (3) holds, which is actually also the main result of this paper, can be stated as follows:

Theorem 1. *Assume that \mathcal{N} is simply connected, i.e. $\mathfrak{p}_c(\mathcal{N}) \neq 1$ and let $1 < p < m$. Then we have*

$$\mathcal{T}_{\text{ext}}^p(\partial\mathcal{M}, \mathcal{N}) = W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N}) \quad (\text{Ext}_p(\mathcal{M}, \mathcal{N})) \quad \{\text{theglaude}\}$$

if and only if

$$p < \mathfrak{p}_c(\mathcal{N}) + 1. \quad (5) \quad \{\text{thecondition}\}$$

We recall that the fact that condition (5) is *sufficient* has already been proved by Hardt and Lin in [10], where a construction of an extension U for any map $u \in W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$ is provided. The main result of this paper is hence the proof that the condition is also *necessary*. This amounts, in the case $m > p > \mathfrak{p}_c(\mathcal{N}) + 1$, to construct a map in $W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$ which cannot be extended as a $W^{1,p}(\mathcal{M}, \mathcal{N})$ map. Several earlier results have already pointed out such obstructions in various examples. For instance, it is shown in [10, 3] that the existence of topological singularities for maps in $W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$ in the case $\pi_{[p-1]}(\mathbb{N}) \neq \{0\}$ provides such obstructions to the extension. In [3], the result of Theorem 1 is proved in the case the target is the circle $\mathcal{N} = \mathbb{S}^1$ (which is of course not simply connected¹). The obstruction there does not involve topological singularities and relies on *lifting* properties of \mathbb{S}^1 -valued maps.

We emphasize that the topology of \mathcal{M} does not enter in the statement, in contrast with the case $p \geq m$ discussed before, for which the topology of \mathcal{M} might be an additional source of obstructions. As a matter of fact, the core of our argument does not involve the topology of the domain and readily deals with the case where $\partial\mathcal{M} \subset \mathbb{R}^{m-1}$, with a map which is constant off the standard ball \mathbb{B}^{m-1} . More precisely, we prove:

Proposition 1. *Assume that $\mathfrak{p}_c(\mathcal{N}) \neq 1$ and that $m_c \equiv \mathfrak{p}_c(\mathcal{N}) + 1 \leq p < m$. There exists a map u_{obst} such that $u_{\text{obst}} = q_0$ on $\mathbb{R}^{m-1} \setminus \mathbb{B}^{m-1}$, where q_0 is an arbitrary point on \mathcal{N} , such that*

$$u_{\text{obst}} - q_0 \in W^{1-1/p,p}(\mathbb{R}^{m-1}, \mathbb{R}^\ell) \quad \text{and} \quad u(x) \in \mathcal{N} \text{ for a.e. } x \in \mathbb{R}^{m-1}, \quad (6)$$

and such that there exist no map U in $W^{1,p}(\mathbb{B}^{m-1} \times [0, 1], \mathcal{N})$ satisfying

$$U(\cdot, 0) = u_{\text{obst}}(\cdot) \text{ on } \mathbb{B}^{m-1} \times \{0\} \text{ in the sense of traces.}$$

Theorem 1 is then deduced in a rather direct way from Proposition 1.

1.3 The case \mathcal{N} is not simply connected

We discuss in this paragraph the case when \mathcal{N} is not simply connected, that is

$$\pi_1(\mathcal{N}) \neq \{0\}. \quad (7)$$

Several results of topological flavor which enter in the proof of Proposition 1 do not extend to the case \mathcal{N} is not simply connected, this is in particular the case for the Hurewicz isomorphism theorem, which is involved in some of our topological arguments. It turns out that the case the manifold \mathcal{N} is simply connected is strongly related to properties and the nature of the universal covering \mathcal{N}_{cov} of \mathcal{N} as well as the lifting property for Sobolev maps. Let

$$\Pi : \mathcal{N}_{\text{cov}} \rightarrow \mathcal{N}$$

denote the covering map. If $\pi_1(\mathcal{N}) = \{0\}$, then $\mathcal{N}_{\text{cov}} = \mathcal{N}$ and Π is the identity. The universal covering is always simply connected, that is $\pi_1(\mathcal{N}_{\text{cov}}) = \{0\}$, so that $2 \leq \mathfrak{p}_c(\mathcal{N}_{\text{cov}}) \leq +\infty$. The simplest example is provided by the case $\mathcal{N} = \mathbb{S}^1$, for which $\pi_1(\mathbb{S}^1) = \mathbb{Z}$. In this example the universal covering is given by $\mathcal{N}_{\text{cov}} = \mathbb{R}$ and hence is not compact. The covering map is the exponential map given by $\Pi(\theta) = \exp i\theta$ for $\theta \in \mathbb{R}$. Another classical example is

¹However the proof of Theorem 1 provided in this paper carries over to this special case, see the discussion in subsection 7

given by the Lie group of rotations of the three-dimensional space $\mathcal{N} = SO(3)$, for which $\pi_1(SO(3)) = \mathbb{Z}_2$. Here the covering space is the group $SU(2)$, which, in contrast to the first example, is compact. As a matter of fact, an important observation is that \mathcal{N}_{cov} is a *compact Riemannian manifold* if and only if $\pi_1(\mathcal{N})$ is a *finite group*.

Given $p > 1$, we say that a map $u \in W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$ is *liftable* if and only if there exists a map $\varphi \in W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N}_{\text{cov}})$ such that

$$u = \Pi \circ \varphi, \tag{8} \quad \{\text{liftable}\}$$

and that property $\mathcal{L}_p^{\text{ift}}(\partial\mathcal{M}, \mathcal{N})$ holds if and only if every map $u \in W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$ is liftable. A first elementary observation which stresses the close relationship between the lifting property $\mathcal{L}_p^{\text{ift}}(\partial\mathcal{M}, \mathcal{N})$ and the extension problem is given in the following result:

Lemma 1. *Assume that $p \geq 2$ and that \mathcal{M} and $\partial\mathcal{M}$ are simply connected. If the extension property $\text{Ext}_p(\mathcal{M}, \mathcal{N})$ holds, then the lifting property $\mathcal{L}_p^{\text{ift}}(\partial\mathcal{M}, \mathcal{N})$ holds also.* {\drouot}

Proof. The proof relies on the fact that the lifting property holds in the space $W^{1,p}(\mathcal{M}, \mathcal{N})$ for $p \geq 2$, that is given an arbitrary map $U \in W^{1,p}(\mathcal{M}, \mathcal{N})$, there exists some $\Phi \in W^{1,p}(\mathcal{M}, \mathcal{N}_{\text{cov}})$ such that $U = \Pi \circ \Phi$ (see e.g. [2] Theorem 1 or [15]). Since we assume that the spaces $\mathcal{T}_{\text{ext}}^p(\partial\mathcal{M}, \mathcal{N})$ and $W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$ coincide, it follows for any map $u \in W^{1-1/p,p}(\mathcal{M}, \mathcal{N})$ there exists a map U in $W^{1,p}(\mathcal{M}, \mathcal{N})$ such that $U = u$ on $\partial\mathcal{M}$. Since $U = \pi \circ \Phi$, it follows in view of the trace Theorem, that

$$u = \pi \circ \varphi,$$

where φ is the trace of Φ on the boundary $\partial\mathcal{M}$, so that u possesses a lifting. The conclusion hence follows. \square

Lemma 1 shows that obstructions to the lifting property yield obstructions to the extension problem. The idea to use obstructions to liftings was introduced first in [3] to prove that in the special case $\mathcal{N} = \mathbb{S}^1$, the answer to the extension problem is negative for $3 \leq p < m$. The obstruction to the lifting property was then generalized in [4] in the general setting of $W^{s,p}(\mathcal{N}, \mathbb{S}^1)$ maps, showing that, turning back to our central problem^{2 3}, obstructions to extensions appear for the exponents $2 \leq p < m$. As matter of fact, this type of obstruction might be generalized to the case the \mathcal{N}_{cov} is not compact, that is when $\pi_1(\mathcal{N})$ is infinite. We have:

Theorem 2. *Assume that $\pi_1(\mathcal{N})$ is infinite. Then the extension property $(\text{Ext}_p(\mathcal{M}, \mathcal{N}))$ does not hold for $2 \leq p < m$.* {\deux}

In other words, the non-existence part⁴ of Theorem 1 remains valid in the case \mathcal{N} is simply connected, provided the fundamental group is infinite. Notice that Theorem 2 does not cover the case $1 \leq p < 2$: This leads to a first open question, namely prove (or disprove) property $\text{Ext}_p(\mathcal{M}, \mathcal{N})$ when

$$(\mathcal{O}1) \quad \pi_1(\mathcal{N}) \text{ is infinite and non trivial} \quad \text{and} \quad 1 \leq p < 2 \leq m.$$

²The may also check that the construction introduced in the proof of Proposition 1 can be carried over to the special case $\mathcal{N} = \mathbb{S}^1$, $2 \leq p < m$, yielding hence an alternate proof

³in the range $2 \leq p < 3$, the construction in [10] yields another obstruction

⁴which, as mentioned, is the main contribution of this paper

Let us actually mention that when $1 \leq p < 2 \leq m$, then the lifting property $\mathcal{L}_p^{\text{ift}}(\partial\mathcal{M}, \mathcal{N})$ holds (see Theorem 3 in [2]). It would be tempting to conclude, in view of the construction of [10] in that case, that the answer is positive. However, since \mathcal{N}_{cov} is not compact, the adaptation of the Hardt-Lin method does not seem straightforward.

We finally turn to the case $\pi_1(\mathcal{N})$ is finite and non trivial. In this case also, we have only partial results. We set

$$\tilde{\mathfrak{p}}_c(\mathcal{N}) = \mathfrak{p}_c(\mathcal{N}) = \inf\{j \in \mathbb{N}^* \setminus \{1\}, \pi_j(\mathcal{N}) \neq \{0\}\}$$

Since the homotopy groups of \mathcal{N}_{cov} of order higher to 2 are equal to the homotopy groups of \mathcal{N} we actually have $\tilde{\mathfrak{p}}_c(\mathcal{N}) = \mathfrak{p}_c(\mathcal{N}_{\text{cov}})$.

{trois}

Theorem 3. *Assume that $\pi_1(\mathcal{N})$ is finite and non trivial.*

i) The extension property $\text{Ext}_p(\mathcal{M}, \mathcal{N})$ does not hold in the following two cases:

- $\tilde{\mathfrak{p}}_c(\mathcal{N}) + 1 \leq p < m$.
- $2 \leq p < 3 \leq m$.

ii) The extension property $(\text{Ext}_p(\mathcal{M}, \mathcal{N}))$ holds if $1 \leq p < 2$.

iii) If $3 \leq p < \tilde{\mathfrak{p}}_c(\mathcal{N}) + 1 < m$, then the extension property $(\text{Ext}_p(\mathcal{M}, \mathcal{N}))$ holds if and only if the lifting property $\mathcal{L}_p^{\text{ift}}(\partial\mathcal{M}, \mathcal{N})$ holds.

In view of the results described in Theorems 1, 2 and 3, the only other case which remains open, when \mathcal{M} is simply connected, corresponds to the case:

$$(\mathcal{O}2) \quad \pi_1(\mathcal{N}) \text{ is finite and non trivial and } 3 \leq p < \tilde{\mathfrak{p}}_c(\mathcal{N}) + 1 < m.$$

Indeed, in this case it follows from Theorem 3 and Lemma 1 that properties $\text{Ext}_p(\mathcal{M}, \mathcal{N})$ and $\mathcal{L}_p^{\text{ift}}(\partial\mathcal{M}, \mathcal{N})$ are *equivalent*. However, to the author's knowledge, the later problem remains completely open in the range of exponents p considered.

Remark 1. The lifting problem $\mathcal{L}_p^{\text{ift}}(\partial\mathcal{M}, \mathcal{N})$ possesses some strong resemblance with the square (or the k -th) root problem for \mathbb{S}^1 valued maps in the Sobolev class. This problem, which was addressed in [2], is *solved* with a positive answer by Mironescu in [11, 12] for $W^{1-1/p, p}(\partial\mathcal{M}, \mathbb{S}^1)$ maps, when $p \geq 3$. The proof relies on an ingenious decomposition of the lifting, somewhat in the same spirit as the one introduced in [5]. These results *might possibly* suggest that the answer to $(\mathcal{O})_2$ is also positive.

Whereas the proofs of Theorems 2 and 3 are essentially combinations of earlier known results (combined with Theorem 1), the main contribution of the present paper is Theorem 1 and its main ingredient Proposition 1. The rest of this introduction presents an outline of its proof.

1.4 On the proof of Proposition 1

Let us first show that the map \mathbf{u}_{obst} constructed in Proposition 1 *cannot be not regular*. In order to get convinced of this fact, we introduce the set

$$\mathcal{T}_{\text{race}, \mathbf{q}_0}^p(\mathcal{N}) = \mathcal{T}_{\text{race}, \mathbf{q}_0}^{m, p}(\mathcal{N}) \equiv \{v \in W^{1-1/p, p}(\mathbb{R}^{m-1}, \mathcal{N}) \text{ with } v = \mathbf{q}_0 \text{ on } \mathbb{R}^{m-1} \setminus \mathbb{B}^{m-1}\} \quad (9) \quad \{\text{tracer}\}$$

and the quantity $\mathfrak{E}_p^{\text{xt}}(u)$ defined for $u \in \mathbb{T}_{\text{race}, q_0}^p(\mathcal{N})$ as

$$\mathfrak{E}_p^{\text{xt}}(u) = \mathfrak{E}_{m,p}^{\text{xt}}(u) = \inf\{E_p(U, \mathcal{D}_m), U \in W_{\text{loc}}^{1,p}(\mathcal{D}_m, \mathcal{N}), U(x, 0) = u(x) \text{ for } x \in \mathbb{R}^{m-1}\}, \quad (10) \quad \{\text{gammap}\}$$

with the convention that the value is infinite when the defining set is empty, where the p -Dirichlet energy E_p is defined for a domain Ω as

$$E_p(v, \Omega) = \int_{\Omega} |\nabla v|^p dx, \text{ for } v : \Omega \rightarrow \mathbb{R}^\ell.$$

and where we have also set

$$\mathcal{D}_m = \mathbb{R}^{m-1} \times [0, 1].$$

In this setting, Proposition 1 can be rephrased as

$$\mathfrak{E}_p^{\text{xt}}(\mathbf{u}_{\text{obst}}) = +\infty. \quad (11) \quad \{\text{rephrased}\}$$

On the other hand, if u belongs to the space $W^{1,p}(\mathbb{B}^{m-1}, \mathcal{N})$ then choosing as a comparison function in (10) the map U defined on \mathcal{D}_m by $U(x, t) = u(x)$ for $x \in \mathbb{R}^{m-1}$, and $t \in [0, 1]$, then we are led to the inequality

$$\mathfrak{E}_p^{\text{xt}}(u) \leq E_p(u).$$

Comparing this inequality with (11), we are led to the conclusion that \mathbf{u}_{obst} *does not belong* to the space $W^{1,p}(\mathbb{B}^{m-1}, \mathcal{N})$ and hence is not Lipschitz. However, Although the map \mathbf{u}_{obst} is not regular, an important intermediate step in the proof of Proposition 1 is to obtain lower bounds on $\mathfrak{E}_p^{\text{xt}}$ for specific lipschitz functions. We define for that purpose for $u \in \mathbb{T}_{\text{race}, q_0}^{m, m_c}$ the quantity

$$\mathcal{I}_m^{\text{xt}}(u) = \inf\{E_{\mathbf{p}_c}(U, C_{\text{yld}}^m(3/2)), U \in \mathfrak{W}_m(u)\}, \quad (12) \quad \{\text{cradoc}\}$$

where we have set

$$\mathfrak{W}_m(u) = \{U \in W_{\text{loc}}^{1, m_c}(C_{\text{yld}}^m(3/2), \mathcal{N}), U(x, 0) = u(x) \text{ for } x \in \mathbb{R}^{m-1}\} \quad (13) \quad \{\text{define}\}$$

and, for $0 \leq r \leq 2$,

$$C_{\text{yld}}^m(r) = \mathbb{B}^{m-1}(r) \times [0, \frac{r}{2}]. \quad (14) \quad \{\text{chapeau}\}$$

Notice that, in definition (12), we choose the exponent for the energy functional $E_{\mathbf{p}_c}$ to be equal to \mathbf{p}_c , whereas the integrability of the test maps U is higher, since it is assumed to be equal to $m_c = \mathbf{p}_c + 1$. We have:

Proposition 2. *Assume that $\mathbf{p}_c(\mathcal{N}) \neq 1$ let m be an integer such that $m \geq \mathbf{p}_c(\mathcal{N})$. For any integer $k \in \mathbb{N}^*$, there exists a Lipschitz map \mathfrak{U}_m^k in $\mathbb{T}_{\text{race}, q_0}^{m, p}$ (\mathcal{N}) such that* \{\text{pirate}\}

$$\begin{cases} \|\nabla \mathfrak{U}_m^k\|_{L^\infty(\mathbb{R}^{m-1})} \leq c_{m,1}k \text{ and} \\ \mathcal{I}_m^{\text{xt}}(\mathfrak{U}_m^k) \geq c_{m,2}k^{\mathbf{p}_c} \geq c_3 E_{\mathbf{p}_c}(\mathfrak{U}_m^k), \end{cases} \quad (15) \quad \{\text{sept}\}$$

where $c_{m,1} > 0$, $c_{m,2} > 0$ and $c_{m,3}$ are constants which do not depend on k .

We next describe some observations which lead to the proof of Proposition 2.

1.4.1 The linear extension operator

Consider first a map in $W^{1-1/p,p}(\mathbb{R}^{m-1}, \mathbb{R}^\ell)$ such that $u = 0$ on $\mathbb{R}^{m-1} \setminus \mathbb{B}^{m-1}$, then the interpolation inequality for the $W^{1-1/p,p}$ norm yields, for some universal constant C_m depending only on m

$$\|u\|_{1-1/p,p} \leq C_m \|u\|_{1,p}^{1-1/p} \|u\|_p^{1/p} \quad \text{provided } u \in W^{1,p}(\mathbb{R}^{m-1}). \quad (16) \quad \{\text{frodonet}\}$$

On the other hand, it follows from standard extension results that there exists a linear operator $\mathcal{T}_{\text{ext}}^p : T_{\text{race},0}^{m,p} \rightarrow W_{m,\text{ct}}^{1,p}$, where

$$\begin{cases} T_{\text{race},0}^{m,p} = T_{\text{race},0}^{m,p}(\mathbb{R}^\ell) \equiv \{v \in W^{1-1/p,p}(\mathbb{R}^{m-1}, \mathbb{R}^\ell) \text{ with } v = 0 \text{ on } \mathbb{R}^{m-1} \setminus \mathbb{B}^{m-1}\} & \text{and} \\ W_{m,\text{ct}}^{1,p} \equiv \{V \in W^{1,p}(\mathbb{R}^{m-1} \times [0,1], \mathbb{R}^\ell) \text{ with } v = 0 \text{ on } (\mathbb{R}^{m-1} \setminus \mathbb{B}^{m-1}(2)) \times [0,1]\}, \end{cases}$$

such that, if $U = \mathcal{T}_{\text{ext}}^p(u)$, then

$$\|U\|_{1,p} = \|\mathcal{T}_{\text{ext}}^p(u)\|_{1,p} \leq C \|u\|_{1-1/p,p}. \quad (17) \quad \{\text{extension}\}$$

Combining (17) with estimate (16) we are led, in case $u \in W^{1,p}(\mathbb{R}^{m-1}, \mathbb{R}^\ell)$ to the estimate

$$\|\nabla \mathcal{T}_{\text{ext}}^p(u)\|_{L^p(\mathcal{D}_m)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^{m-1})}^{1-\frac{1}{p}} \|u\|_{L^p(\mathbb{R}^{m-1})}^{\frac{1}{p}}. \quad (18) \quad \{\text{ineq}\}$$

We turn back to \mathcal{N} -valued maps. Since the manifold \mathcal{N} is compact, we may choose some number L such that $|y| \leq L$ for any $y \in \mathcal{N}$, and hence inequality (18) applies to any Lipschitz map $u \in T_{\text{race},q_0}^p(\mathcal{N})$ yields

$$\|u\|_{1-1/p,p} \leq C_L \|\nabla u\|_{L^\infty(\mathbb{R}^{m-1})}^{1-\frac{1}{p}}. \quad (19)$$

Setting $\Gamma_p^{\text{xt}}(u) = E_p(\mathcal{T}_{\text{ext}}^p(u))$ we are led to the estimate

$$\Gamma_p^{\text{xt}}(u) \leq C (E_p(u))^{1-\frac{1}{p}} \leq C \|\nabla u\|_{L^\infty(\mathbb{R}^{m-1})}^{1-\frac{1}{p}}. \quad (20) \quad \{\text{douze}\}$$

It is worthwhile to compare estimate (20) with the corresponding inequality (15) for the quantite $\mathfrak{E}_p^{\text{xt}}$ for the maps \mathfrak{U}_m^k and to notice the differences in the power laws in term of the energy E_p and the L^∞ norm of the gradient as k grows to $+\infty$.

Remark 2. In [10], Hardt and Lin have succeeded to show that the inequality

$$\mathfrak{E}_p^{\text{xt}}(u) \leq C (E_p(u))^{1-\frac{1}{p}} \quad (21) \quad \{\text{hardtlin}\}$$

holds for $1 \leq p < \mathfrak{p}_c(\mathcal{N}) + 1$, constructing a kind of non linear analog of the operator $\mathcal{T}_{\text{ext}}^p$ which preserves the constraint on the target. Their proof uses a tricky reprojection method. Notice that in the special case u is assumed to be moreover Lipschitz, then (21) yields the estimate

$$\mathfrak{E}_p^{\text{xt}}(u) \leq C \|\nabla u\|_{L^\infty(\mathbb{R}^{m-1})}^{p-1}. \quad (22) \quad \begin{matrix} \{\text{hardtlin2}\} \\ \{\text{carraso}\} \end{matrix}$$

Remark 3. Proposition 1 shows that an inequality similar to (22) does not hold for $\mathfrak{p}_c(\mathcal{N}) + 1 < p < m$. Indeed, we have

$$\mathfrak{E}_p^{\text{xt}}(\mathfrak{U}_m^k) \geq C_m k^p \text{ whereas } \|\nabla \mathfrak{U}_m^k\|_{L^\infty(\mathbb{R}^{m-1})} \leq c_{m,1} k, \quad (23) \quad \{\text{shakelton}\}$$

the first inequality being a consequence of the second inequality in (15) and inequality (3.4) established in subsection 3.1.

In the next paragraph, we will outline the main topological nature of the obstruction to inequality (21) as well as the main ideas in the construction of Proposition 1.

1.4.2 Conservation of topological fluxes

We begin this subsection with a few elementary remarks of topological nature. We start with a general observation concerning the space $C_{q_0}^0(\mathbb{B}^{m-1}, \mathbb{R}^\ell)$ defined by

$$C_{q_0}^0(\mathbb{B}^{m-1}, \mathbb{R}^\ell) = \{w \in C^0(\mathbb{B}^{m-1}, \mathbb{R}^\ell), \text{ s.t } w(x) = q_0 \text{ on } \partial\mathbb{B}^{m-1} \text{ for some } q_0 \in \mathbb{R}^\ell\}.$$

Maps v in $C_{q_0}^0(\mathbb{B}^{m-1}, \mathbb{R}^\ell)$ will be considered sometimes as maps defined on the whole space \mathbb{R}^{m-1} extending their value by $v(x) = q_0$ on $\mathbb{R}^{m-1} \setminus \mathbb{B}^{m-1}$, so that they are still continuous considered as maps on \mathbb{R}^{m-1} . We recall that $C_{q_0}^0(\mathbb{B}^{m-1}, \mathbb{R}^\ell)$ may be mapped one to one to the space $C^0(\mathbb{S}^{m-1}, \mathbb{R}^\ell)$ thanks to the stereographic projection St_{m-1} which is a smooth map from $\mathbb{S}^{m-1} \setminus \{P_{\text{south}}\} \subset \mathbb{R}^m$ onto \mathbb{R}^{m-1} and is defined by

$$\text{St}_{m-1}(x_1, \dots, x_m) = \left(\frac{x_1}{1+x_m}, \dots, \frac{x_{m-1}}{1+x_m} \right),$$

with $P_{\text{south}} = (0, 0, 0, \dots, -1)$. It follows that given any map v in $C_{q_0}^0(\mathbb{R}^{m-1}, \mathcal{N})$ the map $v \circ \text{St}_{m-1}^{-1}$ belongs to $C^0(\mathbb{S}^{m-1}, \mathcal{N})$. This allows to identify maps in $C_{q_0}^0(\mathbb{B}^{m-1}, \mathcal{N})$ with maps in $C^0(\mathbb{S}^{m-1}, \mathcal{N})$. Moreover, we have a one to one correspondance of homotopy classes. Given a map $\varphi \in C_{q_0}^0(\mathbb{B}^{m-1}, \mathcal{N})$ we denote by $[\![\varphi]\!]$ its homotopy class.

We consider next a map $V \in C^0(\mathbb{R}^{m-1} \times [0, 1], \mathcal{N})$ and the map v defined on \mathbb{R}^{m-1} by $v(x) = V(x, 0)$ for $x \in \mathbb{R}^{m-1}$. We assume furthermore that

$$v = q_0 \text{ on } \mathbb{R}^{m-1} \setminus \mathbb{B}^{m-1} \text{ so that } v|_{\mathbb{B}^{m-1}} \in C_{q_0}^0(\mathbb{B}^{m-1}, \mathcal{N}), \quad (24) \quad \{\text{confitde}\}$$

For $1 \leq r \leq 2$, consider the cylinder $C_{\text{yld}}(r) = C_{\text{yld}}^m(r)$, with C_{yld}^m defined in (14) and denote by $\Lambda^{m-1}(r)$ the inner part of the boundary defined by

$$\Lambda^{m-1}(r) = \left(\partial\mathbb{B}^{m-1}(r) \times [0, \frac{r}{2}] \right) \cup \mathbb{B}^{m-1}(r) \times \left\{ \frac{r}{2} \right\} \text{ so that } \partial C_{\text{yld}}^m(r) = \Lambda_r^{m-1} \cup \mathbb{B}^{m-1}(r) \times \{0\}.$$

Notice that $\Lambda^{m-1}(r)$ may be mapped homeomorphically to the ball \mathbb{B}_r^{m-1} by a bilipschitz homeomorphism Φ_r whose Lipschitz constants may be bounded independently of r , that is $\|\nabla \Phi_r\|_\infty + \|\nabla \Phi_r^{-1}\|_\infty \leq C$. Since, in view of (24), the restriction of the map V to $\partial\Lambda_r^{m-1} = \partial\mathbb{B}_r^{m-1} \times \{0\}$ is constant, we may define the homotopy class of its restriction to Λ_r which we oriente according to the outer normal to ∂C_{yld} . We claim that, with this choice of orientation we have

$$[\![V|_{\Lambda^{m-1}(r)}]\!] = -[\![v|_{\mathbb{B}^{m-1}}]\!] \text{ for any } 1 \leq r \leq 2. \quad (25) \quad \{\text{claim}\}$$

Indeed, since V is continuous inside the cylinder $C_{\text{yld}}(r)$, its restriction to the boundary $\partial C_{\text{yld}}(r)$, which is homeomorphic to the sphere \mathbb{S}^{m-1} , has trivial homotopy class. On the other hand we have

$$[\![V|_{\partial C_{\text{yld}}(r)}]\!] = [\![V|_{\Lambda^{m-1}(r)}]\!] + [\![v|_{\mathbb{B}^{m-1}}]\!]$$

so that the conclusion (25) follows. The identity (25) extends to Sobolev maps, provided the exponent p is larger than m . We have:

Lemma 2. Assume that $p \geq m$ and let $v \in T_{\text{face}, \mathbf{q}_0}^p(\mathcal{N})$ and $V \in W_{\text{loc}}^{1,p}(\mathcal{D}_m, \mathcal{N})$ be such that

$$V(\cdot, 0) = v(\cdot) \text{ in the sense of traces on } \mathbb{R}^{m-1}. \quad (26)$$

Then, the homotopy classes $\llbracket v|_{\mathbb{B}^{m-1}} \rrbracket$ and $\llbracket V|_{\Lambda^{m-1}(r)} \rrbracket$ are well defined for every $1 < r \leq 2$ and moreover (25) holds.

The proof is immediat for $p > m$, since in that case V is continuous by Sobolev embedding. The limiting case $p = m$ requires more care and follows adapting ideas from the works of Brezis and Nirenberg [6, 7].

Remark 4. The result of Lemma 2 *does not hold* when $1 \leq p < m$, due to the possibility of having *topological* singularities. Assume indeed that $\pi_{m-1}(\mathcal{N}) \neq \{0\}$ and consider a map in $v \in C_{\mathbf{q}_0}^0(\mathbb{B}^{m-1}, \mathbb{R}^\ell)$ having non trivial homotopy class and extended outside \mathbb{B}^{m-1} by \mathbf{q}_0 . Let $Q = (0, \dots, 0, \frac{1}{2}) \in \mathbb{R}^m$. Given a point $M = (x_1, \dots, x_{m-1}, x_m) \in \mathbb{R}^{m-1} \times [0, 1]$ we set

$$\begin{cases} V(M) = v(\Phi(M)) \text{ if } x_m > \frac{1}{2} \text{ where } \Phi(M) = D(Q, M) \cap \mathbb{R}^{m-1} \times \{0\} \\ V(M) = \mathbf{q}_0 \text{ otherwise,} \end{cases} \quad (27)$$

where $D(Q, M)$ denotes the line joining Q to M . It follows that, if $p < m$, then $v \in W^{1,p}(\mathcal{D}^m, \mathcal{N})$ with $V = v$ on $\mathbb{B}^{m-1} \times \{0\}$, the map V being continuous except at the point Q , where it has a singularity carrying a topological charge (the restriction to any small sphere around Q has non trivial topology). On the other hand, we have, for $1 \leq r \leq 2$

$$V(M) = \mathbf{q}_0 \text{ for } M \in \Lambda_r^{m-1} \text{ so that } \llbracket V|_{\Lambda^{m-1}(r)} \rrbracket = \{0\},$$

and hence $\llbracket V|_{\Lambda^{m-1}(r)} \rrbracket \neq -\llbracket v|_{\mathbb{B}^{m-1}} \rrbracket$. Notice that the map V no longer belongs to $W^{1,p}$ when $p \geq m$.

In the next section, we will see how these *topological fluxes* through the sets Λ_r^{m-1} generate also *energy fluxes*.

1.4.3 Infimum of energy in homotopy classes and energy fluxes

For an integer $\mathbf{n} \geq 2$ and an exponent $p \geq 1$ and a map $\varphi \in C_{\mathbf{q}_0}^0(\mathbb{B}^{\mathbf{n}}, \mathcal{N})$, we consider the numbers

$$\mathbf{v}_{\mathbf{n},p}(\llbracket \varphi \rrbracket) = \inf\{E_p(w), w \in \text{Lip}_{\mathbf{q}_0}(\mathbb{B}^{\mathbf{n}}, \mathcal{N}) \text{ homotopic to } \varphi\}.$$

It follows from the scaling law for the energy

$$E_p(u_r, \mathbb{B}^{\mathbf{n}}(r)) = r^{\mathbf{n}-p} E_p(u, \mathbb{B}^{\mathbf{n}}) \text{ where } u_r(x) = u(rx) \text{ for } x \in \mathbb{B}^{\mathbf{n}}, E_p(v_r) \quad (28)$$

that, for any $1 \leq p < \mathbf{n}$, we have (letting r go to zero in the above identity)

$$\mathbf{v}_{\mathbf{n},p}(\llbracket \varphi \rrbracket) = 0 \text{ for any homotopy class } \llbracket \varphi \rrbracket,$$

whereas

$$\text{when } p \geq \mathbf{n}, \text{ then } \mathbf{v}_{\mathbf{n},p}(\llbracket \varphi \rrbracket) = 0 \text{ if and only if } \llbracket \varphi \rrbracket = 0.$$

Going back to Lemma 2 and invoking scale invariance, we obtain a lower bound for the energy on surfaces $\Lambda(r)$, in the special case $m = m_c$, as stated in the next result.

Proposition 3. *Assume that $p \geq m$ and that v and V are as in Lemma 2. Given $p \geq s \geq m - 1$, we have, for every $r \in [1, 2]$ and some constant $C_s > 0$* {surfaces}

$$\int_{\Lambda^{m-1}(r)} |\nabla v|^s \geq C_s \mathbf{v}_{m-1,s}(\llbracket v \rrbracket). \quad (29) \quad \{\text{micromou}\}$$

As a matter of fact, we will mainly invoke this inequality with the exponent $s = m - 1$, so that we are led to introduce the numbers

$$\mathbf{v}_{\mathbf{n}}(\llbracket v \rrbracket) \equiv \mathbf{v}_{\mathbf{n},\mathbf{n}}(\llbracket v \rrbracket) \text{ for } \mathbf{n} \in \mathbb{N}^*. \quad (30) \quad \{\text{matou}\}$$

Combining Hölder's inequality with (29) we obtain for V as in Lemma 2

$$\int_{\Lambda^{m-1}(r)} |\nabla v|^s \geq C_s [\mathbf{v}_{m-1}(\llbracket v \rrbracket)]^{\frac{s}{m-1}} \text{ for } 1 \leq r \leq 2. \quad (31) \quad \{\text{micromou2}\}$$

We discuss next some specific properties of the numbers $\mathbf{v}_{m-1}(\llbracket \varphi \rrbracket)$ in the special case

$$m = m_c \equiv \mathbf{p}_c(\mathcal{N}) + 1. \quad (32) \quad \{\text{dimcritic}\}$$

when $\mathbf{p}_c \geq 2$. In that case, the manifold \mathcal{N} is $(\mathbf{p}_c - 1)$ -connected⁵, a fact which has important consequences on the relevant homotopy group $\pi_{\mathbf{p}_c}(\mathcal{N})$. Such manifolds possess indeed some strong similarities with joints of spheres \mathbb{S}^q . In particular, the homotopy group $\pi_{\mathbf{p}_c}(\mathcal{N})$ is finitely generated and, if $\sigma_1, \dots, \sigma_{\mathfrak{s}}$ denote the generators of $\pi_{\mathbf{p}_c}(\mathcal{N})$, then the sub-groups generated by each of the σ_i 's is infinite. For $d \in \mathbb{Z}$ we set, denoting by \star the composition law in $\pi_{\mathbf{p}_c}(\mathcal{N})$,

$$\sigma_i^d = \underbrace{\sigma_i \star \dots \star \sigma_i}_{d \text{ times}}.$$

Proposition 4. *Assume that $\mathbf{p}_c \geq 2$. There exists constants $C_1 > C_2 > 0$ depending only on \mathcal{N} such that for any $i = 1, \dots, \mathfrak{s}$, we have* {clefkey}

$$C_1 |d| \geq \mathbf{v}_{\mathbf{p}_c}(\sigma_i^d) \geq C_2 |d|. \quad (33) \quad \{\text{troc}\}$$

Moreover, for every $i = 1, \dots, \mathfrak{s}$ and every $d \in \mathbb{Z}$ there exists a Lipschitz map \mathbf{v}_d^i from $\mathbb{B}^{\mathbf{p}_c}$ to \mathcal{N} such that $\llbracket \mathbf{v}_d^i \rrbracket_i = \sigma_i^d$,

$$|\nabla \mathbf{v}_d^i|^{\mathbf{p}_c} \leq c_0 |d| \text{ in } \mathbb{B}^{\mathbf{p}_c} \text{ and } \mathbf{v}_d^i = \mathbf{q}_0 \text{ on } \partial \mathbb{B}^{\mathbf{p}_c}, \quad (34) \quad \{\text{gluts}\}$$

where $c_0 > 0$ depends only on \mathcal{N} and where $\mathbf{q}_0 \in \mathbb{N}$ is an arbitrary choosen point on \mathcal{N} .

As a matter of fact, in the case $\mathcal{N} = \mathbb{S}^p$, for which $\mathbf{p}_c = p$, the results in Proposition 4 may be deduced directly from degree theory, whereas in the general case, we rely on some more sophisticated notions of topology, in particular related to the theory of CW-complexes.

⁵recall that a manifold is said to be $\mathbf{q} - 1$ connected if $\pi_j(\mathbb{N}) = \{0\}$ for every integer $0 \leq j < \mathbf{q}$.

1.4.4 On the construction of \mathfrak{U}_m^k

We start describing the construction in the case $m = m_c = p_c + 1$, which is actually the building block of the general case. In that case, the construction follows directly from the construction in Proposition 4 since we set, for $k \in \mathbb{N}$

$$\mathfrak{U}_{m_c}^k \equiv \mathfrak{v}_d^i \text{ with } d = k^{p_c}. \quad (35) \quad \{\text{prems}\}$$

It turns out that, as a direct consequence of Proposition 4 and of Proposition 3, that the map $\mathfrak{U}_{m_c}^k$ satisfies assumption (15) for any $k \in \mathbb{N}^*$, provided the constants $c_{p_c,1}$ is chosen sufficiently large and the constant $c_{p_c,2}$ are chosen sufficiently small, a more precise statement being provided in Lemma 3.2. The case $m > m_c = p_c + 1$ is deduced from the construction in the critical dimension $m = m_c = p_c + 1$ adding in a suitable way dimensions.

1.4.5 On the construction of $\mathfrak{u}_{\text{obst}}$

The map $\mathfrak{u}_{\text{obst}}$ is constructed gluing an *infinite* but *countable* number of scaled and translated copies of the maps \mathfrak{U}_m^k , for suitable choices of diverging indices k and shrinking scaling factors. The construction relies in an essential way on two properties. The first one is related to the difference, for the maps \mathfrak{U}_m^k , of the asymptotic behaviors as k grows of the infimum of the energy of the extensions on one hand and the p -th power of trace norm on the other. More precisely, we use extensively the fact that

$$\|\mathfrak{U}_m^k - \mathfrak{q}_0\|_{1-\frac{1}{p},p}^p \leq C_m k^{p-1} \text{ whereas } \mathfrak{E}_{m,p}^{\text{xt}}(\mathfrak{U}_m^k) \geq \mathcal{E}_{m,p}^{\text{xt}}(\mathfrak{U}_m^k) \geq C_m k^p, \quad (36) \quad \{\text{cruxitude}\}$$

where the quantity $\mathcal{E}_{m,p}^{\text{xt}}(u)$, which is localized version of $\mathfrak{E}_{m,p}^{\text{xt}}$, is defined in (3.1). The second important property on which the construction is based upon is related again on scaling properties of the energy functional E_p . It may an may be stated, for a general map $u : \mathbb{R}^m \rightarrow \mathcal{N}$ and $0 < r < 2$ as the identity, similar to (28), namely

$$E_p(u_r, C_{\text{yld}}^m(r)) = r^{m-p} E_p(u, C_{\text{yld}}^m) \text{ where } u_r(x) = u(rx) \text{ for } x \in C_{\text{yld}}^m, \quad (37) \quad \{\text{scalingprop1}\}$$

so that in particular $E_p(u_r, C_{\text{yld}}^m(r))$ tends to 0 as the scaling factor r goes to zero. The scaling law (37) has a counterpart for the semi-norm $|\cdot|_{1-1/p,p}$ given by the relation

$$|u_r|_{1-1/p,p}^p = r^{m-p} |u|_{1-1/p,p}^p \text{ for } u : \mathbb{R}^{m-1} \rightarrow \mathbb{R}^\ell. \quad (38) \quad \{\text{scalsemi}\}$$

The gluing process. We define first the set of points $\{\mathfrak{M}_i\}_{i \in \mathbb{N}}$ in \mathbb{R}^{m-1} where the copies of the maps \mathfrak{U}_m^k will be glued by

$$\mathfrak{M}_i = \left(\sum_{j=0}^i \delta_j \right) \vec{e}_1 \text{ where } \vec{e}_1 = (1, \dots, 0) \in \mathbb{R}^{m-1}, \text{ for } i \in \mathbb{N}, \quad (39) \quad \{\text{toujoursitude}\}$$

and where we have set

$$\delta_i = \frac{1}{a_0 i (\log i)^2} \text{ for } i \in \mathbb{N}^*, \text{ with } a_0 = 2 \sum_{j=0}^{+\infty} \frac{1}{j (\log j)^2} < +\infty.$$

It follows that the points \mathfrak{M}_i are all on the segment joining the origin to the point

$$\mathfrak{M}_\star = \frac{1}{2}\vec{e}_1 = (\frac{1}{2}, 0, \dots, 0),$$

converging to the point \mathfrak{M}_\star as $i \rightarrow +\infty$. We then consider a sequence of radii $(\mathfrak{r}_i)_{i \in \mathbb{N}}$ such that $0 < \mathfrak{r}_i < \frac{1}{4} \inf\{\delta_i, \delta_{i-1}\}$ and the corresponding collection of disjoint balls $(B_i)_{i \in \mathbb{N}}$ given by

$$B_i \equiv \mathbb{B}^{m-1}(\mathfrak{M}_i, \mathfrak{r}_i) \text{ for } i \in \mathbb{N}, \text{ so that } \text{dist}(B_i, B_j) \geq \frac{1}{2} \sup\{\delta_i, \delta_j\} \text{ and } \bigcup_{i \in \mathbb{N}} B_i \subset \mathbb{B}^{m-1}.$$

We finally introduce a sequence of integers $(k_i)_{i \in \mathbb{N}}$ and define the map $\mathfrak{u}_{\text{obst}}$ on \mathbb{R}^{m-1} as

$$\mathfrak{u}_{\text{obst}}(x) = \mathfrak{U}_m^{k_i} \left(\frac{x - \mathfrak{M}_i}{\mathfrak{r}_i} \right) \text{ if } x \in B_i, \quad \mathfrak{U}(x) = \mathfrak{q}_0 \text{ if } x \in \mathbb{R}^{m-1} \setminus \bigcup_{i \in \mathbb{N}} B_i. \quad (40) \quad \{\text{mathcal{ititude}}\}$$

The next two results, which are directly connected to the scaling laws (37) and (38) reduce the construction of $\mathfrak{u}_{\text{obst}}$ to the search of appropriate sequences $(\mathfrak{r}_i)_{i \in \mathbb{N}}$ and $(k_i)_{i \in \mathbb{N}}$. The first deals with the trace semi-norm of $\mathfrak{u}_{\text{obst}}$.

$\{\text{thrace}\}$

Lemma 3. *Assume that*

$$\sum_{i \in \mathbb{N}} k_i^{p-1} \mathfrak{r}_i^{m-p} < +\infty \text{ and } \mathfrak{r}_i \leq \frac{1}{16} \delta_i. \quad (41) \quad \{\text{hypothesitude}\}$$

Then the map $\mathfrak{u}_{\text{obst}}$ defined in (40) belongs to $\mathcal{T}_{\text{race}, \mathfrak{q}_0}^{m,p}(\mathcal{N})$.

The second result concerns the energy of the extension.

$\{\text{lemmitude}\}$

Lemma 4. *Assume that $m_c \equiv \mathfrak{p}_c(\mathcal{N}) + 1 \leq p < m$. Then we have*

$$\mathfrak{E}_{m,p}^{\text{xt}}(\mathfrak{u}_{\text{obst}}) \geq \sum_{i \in \mathbb{N}} k_i^p \mathfrak{r}_i^{m-p}. \quad (42) \quad \{\text{labelitude}\}$$

The proof of Proposition 1 is then completed by showing that there exists sequences $(\mathfrak{r}_i)_{i \in \mathbb{N}}$ and $(k_i)_{i \in \mathbb{N}}$ such that (41) holds and such that

$$\sum_{i \in \mathbb{N}} k_i^p \mathfrak{r}_i^{m-p} = +\infty. \quad (43) \quad \{\text{moyennitude}\}$$

The fact that this is possible is related to the different exponents for k_i ($p-1$ in the first one and p in the second) in both inequality, a property which ultimately goes back to (36).

1.5 Outline of the paper

$\{\text{outline}\}$

This paper is organized as follows. In the next Section we describe the relationship between energy estimates and topological invariants, in the case the exponent for the energy integral equals the dimension. In particular, we provide the proof to Proposition 4. Section 3 is devoted to the the proof of Proposition 2, whereas the proof to Proposition 1 is given in Section 4. The proofs of the main theorems are finally completed in Sections 5 and 6.

2 Topology and energy estimates

{topenergy}

The main purpose of this section is to provide the proof of Proposition 4. We split it into two parts, each of which corresponds to one of the two statements of the proposition, which require however different assumptions. the main focus is on the numbers $\mathfrak{v}_{\mathfrak{p}}(\llbracket v \rrbracket)$ defined in (30). We start the analysis with an explicit upper bound.

2.1 An upper bound for the energy in homotopy classes

Let $\mathfrak{p} \in \mathbb{N}^*$. We assume throughout this subsection that the \mathfrak{p} -th homotopy group of \mathcal{N} is non trivial that is $\pi_{\mathfrak{p}}(\mathcal{N}) \neq \{0\}$ and that it is infinite. More precisely, we assume that there are elements $\sigma_1, \dots, \sigma_{\mathfrak{s}}$ in $\pi_{\mathfrak{p}}(\mathcal{N})$ such that the sub-group \mathfrak{G}_i generated by σ_i is infinite, that is

$$\mathfrak{G}_i = \{\sigma_i^\ell, \ell \in \mathbb{Z}\} \sim \mathbb{Z}. \quad (2.1)$$

{sim}
{upperb}

Lemma 2.1. *Assume that (2.1) holds. There exists a constant $c_1 > 0$, such that given any $i = 1, \dots, \mathfrak{s}$ and given any $d \in \mathbb{Z}$, there exists a map $\Phi_d^i \in C_{\mathfrak{q}_0}^1(\mathbb{B}^{\mathfrak{p}}, \mathcal{N})$ such that $\llbracket \Phi_d^i \rrbracket = \sigma_i^d$ and*

$$|\nabla \Phi_d^i|(x)^{\mathfrak{p}} \leq c_0 |d|, \text{ for any } x \in \mathbb{B}^{\mathfrak{p}}. \quad (2.2)$$

{labo}

Proof. We start with the case $d = 1$. Given $i = 1, \dots, \mathfrak{s}$ we choose an arbitrary map $\Phi^i = \Phi_1^i \in C_{\mathfrak{q}_0}^1(\mathbb{B}^{\mathfrak{p}}, \mathcal{N})$ such that $\llbracket \Phi^i \rrbracket = \sigma_i$ and set

$$c_1 = \|\nabla \Phi^i\|_{L^\infty(\mathbb{B}^{\mathfrak{p}})} < +\infty. \quad (2.3)$$

{delco}

It follows that (2.2) is fulfilled in the case $d = 1$, provided $c_0 \geq c_1$. We next turn to the case $d \geq 1$. We introduce the set of indices

$$A_{\mathfrak{p}}(d) = \{I = (i_1, i_2, \dots, i_{\mathfrak{p}}), i_k \in \mathbb{N}^*, (i_k)^{\mathfrak{p}} \leq d\},$$

so that the total number of elements in $A_{\mathfrak{p}}(d)$ is given by $\sharp(A_{\mathfrak{p}}(d)) = \left\lceil d^{\frac{1}{\mathfrak{p}}} \right\rceil^{\mathfrak{p}}$, where for $t \in \mathbb{R}^+$, the symbol $\lceil t \rceil$ denotes the largest integer less of equal to t . Notice that, by a convexity argument, we have

$$d - \mathfrak{p}d^{1-\frac{1}{\mathfrak{p}}} \leq \sharp(A_{\mathfrak{p}}(d)) \leq d \text{ so that } 0 \leq r_d \equiv d - \sharp(A_{\mathfrak{p}}(d)) \leq \mathfrak{p}d^{1-\frac{1}{\mathfrak{p}}} < \sharp(A_{\mathfrak{p}}(d)),$$

where the last inequality holds provided d is sufficiently large. We consider a subset $B_{\mathfrak{p}}(d)$ of r_d distinct elements in $A_{\mathfrak{p}}(d)$. We introduce the set of points $\Upsilon = \Upsilon_A \cup \Upsilon_B$, where $\Upsilon_A \equiv \{a_I\}_{I \in A_{\mathfrak{p}}(d)}$ and $\Upsilon_B = \{b_I\}_{I \in B_{\mathfrak{p}}(d)}$, the points a_I and b_I being defined, setting $h = d^{-\frac{1}{\mathfrak{p}}}$ by

$$a_I = \frac{h}{4}I \text{ for } I \in A_{\mathfrak{p}}(d) \text{ and } b_I = \frac{h}{4}I + \left(\frac{1}{2}, \dots, 0\right) \text{ for } I \in B_{\mathfrak{p}}(d),$$

so that the mutual distance between distinct points in Υ is at least $\frac{h}{4}$ and $\sharp\Upsilon = d$. We then define the map Φ_d^i as

$$\begin{cases} \Phi_d^i(x) = \Phi^i\left(\frac{x - a_I}{8h}\right) \text{ for } x \in \mathbb{B}^{\mathfrak{p}}(a_I, \frac{h}{8}), I \in A_{\mathfrak{p}}(d) \\ \Phi_d^i(x) = \Phi^i\left(\frac{x - b_I}{8h}\right) \text{ for } x \in \mathbb{B}^{\mathfrak{p}}(b_I, \frac{h}{8}), I \in B_{\mathfrak{p}}(d) \\ \Phi_d^i(x) = \mathfrak{q}_0 \text{ otherwise.} \end{cases} \quad (2.4)$$

{defphi}

Since Φ_d^i is obtained gluing d scaled copies of Φ^i its homotopy class is σ_i^d , whereas combining (2.3) with (2.4) we obtain (2.2) choosind $c_0 = 8c_1$. This establishes the theorem for $d > 0$. The proof is similar for $d < 0$. \square

Integrating the bound (2.2) on \mathbb{B}^p and using the function Φ_d^i as a test function in the definition (30) of $\nu_p(\sigma_i^d)$ we are led to the upper bound

$$\nu_p(\sigma_i^d) \leq C_2|d|, \text{ for any } d \in \mathbb{Z}, \quad (2.5) \quad \{\text{gratis}\}$$

where $C_2 > 0$ is some constant which does not depend on d . This upper bound actually corresponds to the right part of inequality (33) and, as seen above, this inequality does only require the subgroup \mathfrak{G}_i to be infinite. A natural question is to determine whether there exists also in that case a lower bound of the same magnitude, i.e. to know if there exists a constant $C_i > 0$ such that

$$\nu_p(\sigma_i^d) \geq C_i|d|. \quad (2.6) \quad \{\text{cassegain}\}$$

Such a lower bound can be established for instance if $\mathcal{N} = \mathbb{S}^p$ using degree theory. More precisely, in the case of the sphere \mathbb{S}^p , we have $\pi_p(\mathbb{S}^p) = \mathbb{Z}$, the unique generator of this homotopy group being the homotopy class of the identity. In this case, the degree labels the order in the homotopy group. It is given by the integral formula

$$\deg u = \int_{\mathbb{B}^p} u^*(\omega) d\sigma. \quad (2.7) \quad \{\text{formula}\}$$

where ω is a normalized volume form of the sphere and $*$ denotes pull-back. Formula (2.7) yields rather directly to the upper bound (2.6), in view of the pointwise inequality $|u^*(\omega)| \leq C|\nabla u|^p$. It turns out however that the bound (2.6) does not hold for general manifolds, even if (2.1) holds. This was proved for instance in [16] for the case $p = 3$ and $\mathcal{N} = \mathbb{S}^2$ for which $\pi_3(\mathbb{S}^2) = \mathbb{Z}$. It is shown there that $\nu_p(\sigma^d) \leq C|d|^{\frac{3}{4}}$, which contradicts (2.6) for large values of $|d|$.

2.2 A lower bound for the energy in homotopy classes

$\{\text{lowerenergy}\}$

In view of the previous remark and in order to address the bound (2.6), we need to impose additional conditions on \mathcal{N} . In this subsection, we assume that $p \in \mathbb{N}^* \setminus \{1\}$ and impose that the manifold \mathcal{N} is $(p-1)$ -connected, that is we assume throughout that

$$\pi_1(\mathcal{N}) = \dots = \pi_{p-1}(\mathcal{N}) = \{0\} \text{ and } \pi_p(\mathcal{N}) \neq \{0\}. \quad (2.8) \quad \{\text{souple}\}$$

This kind of assumption is for instance central in the statement of the Hurewicz isomorphism theorem and has also been used in the context of Sobolev maps in several places in the literature (see e.g. [10, 9, 14, 15] among others). The main feature which is used there is that $(p-1)$ -connected manifolds possess strong analogies with the sphere \mathbb{S}^p , or more precisely with joints of p -dimensional spheres. In particular the homotopy group has a finite number of generators $\sigma_1, \dots, \sigma_s$ verifying (2.1), corresponding to each of the spheres. The lower bound for the p -energy of \mathbb{S}^p -valued maps can be generalized to $(p-1)$ -connected manifolds as follows:

{ouistiti}

Lemma 2.2. *Assume that (2.8) holds. Then $\pi_{\mathfrak{p}}(\mathcal{N})$ is infinite. Moreover, if σ_1 is a generator such that (2.1) holds, then there exists a constant $C_i > 0$ such that, for any $d \in \mathbb{Z}$, we have*

$$\mathfrak{v}_{\mathfrak{p}}(\sigma_i^d) \geq C_i |d|. \quad (2.9) \quad \{\text{gratuitude}\}$$

Let us emphasize that this result is *not new* and is actually presumably well-known to the experts. As a matter of fact, the result of Lemma 2.2 can be directly deduced as a special case of Lemma 4.3 in [15]. For sake of completeness however, we briefly explain the main ideas in the proof.

Sketch of the proof (following [15, 9]). The proof relies on several observations, the first ones being related topological properties of the manifold \mathcal{N} we describe next.

Topological background. We consider some smooth triangulation T of \mathcal{N} and denote by \mathcal{N}^j the j -dimensional skeleton of \mathcal{N} for $1 \leq j \leq \nu = \dim \mathcal{N}$, so that $\mathcal{N}^\nu = \mathcal{N}$. It turns out that, if \mathcal{N} is $(\mathfrak{p} - 1)$ -connected, then necessarily one has $\mathfrak{p} \leq \nu$ and the \mathfrak{p} -skeleton $\mathcal{N}^{\mathfrak{p}}$ of \mathcal{N} has the homotopy type of a joint of \mathfrak{s} spheres. Moreover, the \mathfrak{p} -homotopy groups of \mathcal{N} and $\mathcal{N}^{\mathfrak{p}}$ coincide. We have therefore

$$\mathcal{N}^{\mathfrak{p}} \sim \bigvee_{i=1}^{\mathfrak{s}} \mathbb{S}^{\mathfrak{p}} \text{ and } \pi_{\mathfrak{p}}(\mathcal{N}^{\mathfrak{p}}) = \pi_{\mathfrak{p}}(\mathcal{N}).$$

We denote by $\tilde{\sigma}_i, \dots, \tilde{\sigma}_{\mathfrak{s}}$ the generators of $\mathcal{N}^{\mathfrak{p}}$ which also correspond to generators of $\pi_{\mathfrak{p}}(\mathcal{N})$, and set, for $\varphi \in C^0(\mathbb{S}^{\mathfrak{p}}, \mathcal{N}^{\mathfrak{p}})$

$$\langle \varphi \rangle_{i,\mathfrak{p}} = d_i \text{ if } \llbracket \varphi \rrbracket = \tilde{\sigma}_1^{d_1} \star \tilde{\sigma}_2^{d_2} \star \dots \star \tilde{\sigma}_i^{d_i} \star \dots \star \tilde{\sigma}_{\mathfrak{s}}^{d_{\mathfrak{s}}}. \quad (2.10) \quad \{\text{decsigma}\}$$

Properties of maps in $W^{1,\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}, \mathcal{N}^{\mathfrak{p}})$. We restrict ourselves for the moment to maps which take values on the \mathfrak{p} -skeleton $\mathcal{N}^{\mathfrak{p}} \subset \mathcal{N}$, and show that for such a target the lower bound (2.9) holds. Given $i = 1, \dots, \mathfrak{s}$, it can be proved that there exists a smooth "projection" map $\Pi_i : \mathcal{N}^{\mathfrak{p}} \rightarrow \mathbb{S}^{\mathfrak{p}}$, with the property that, if φ is a continous map from $\mathbb{S}^{\mathfrak{p}}$ to $\mathcal{N}^{\mathfrak{p}}$, then $\Pi_i \circ \varphi \in C^0(\mathbb{S}^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$ with

$$\deg(\Pi_i \circ \varphi) = \langle \varphi \rangle_{i,\mathfrak{p}} \text{ for all } \varphi \in C^0(\mathbb{S}^{\mathfrak{p}}, \mathcal{N}^{\mathfrak{p}}). \quad (2.11) \quad \{\text{degradation}\}$$

If φ belongs moreover to the space $W^{1,\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}, \mathcal{N}^{\mathfrak{p}})$, then we have, since Π_i is smooth, the pointwise inequality $|\nabla(\Pi_i \circ \varphi)| \leq C |\nabla \varphi|$, so that

$$E_{\mathfrak{p}}(\Pi_i \circ \varphi) \leq C E_{\mathfrak{p}}(\varphi). \quad (2.12) \quad \{\text{nounours}\}$$

On the other hand, since (2.9) holds for $\mathbb{S}^{\mathfrak{p}}$ -valued maps thanks to degree theory, we have, in view of (2.11)

$$E_{\mathfrak{p}}(\Pi_i \circ \varphi) \geq C |\deg(\Pi_i \circ \varphi)| \geq C |\langle \varphi \rangle_{i,\mathfrak{p}}|$$

so that, combining with (2.12), we obtain, for some constant $C > 0$,

$$E_{\mathfrak{p}}(\varphi) \geq C \left| \langle \varphi \rangle_{i,\mathfrak{p}} \right| \text{ for every } \varphi \in W^{1,\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}, \mathcal{N}^{\mathfrak{p}}). \quad (2.13) \quad \{\text{pimprenelle}\}$$

Projecting onto the \mathfrak{p} -skeleton \mathcal{N}^k . This step corresponds to an adaptation of reprojection method introduced in [10], used for each of the individual simplexes of the triangulation T . This construction yields, for a given map $u \in \text{Lip}(\mathbb{S}^{\mathfrak{p}}, \mathcal{N})$, the existence of another map $\tilde{u} \in \text{Lip}(\mathbb{S}^{\mathfrak{p}}, \mathcal{N}^{\mathfrak{p}})$ such that, for some constant $C > 0$ independent of u

$$E_{\mathfrak{p}}(\tilde{u}) \leq CE_{\mathfrak{p}}(u) \text{ and } \langle \tilde{u} \rangle_{i, \mathfrak{p}} = \langle u \rangle_i \text{ for every } i = 1, \dots, \mathfrak{s} \quad (2.14) \quad \{\text{hhh}\}$$

and, moreover, if $u(x) \in \mathcal{N}^{\mathfrak{p}}$ for some $x \in \mathbb{S}^{\mathfrak{p}}$, then we have $\tilde{u}(x) = u(x)$. In (2.14), we have set similar to (2.10)

$$\langle u \rangle_i = d_i \text{ if } \llbracket u \rrbracket = \sigma_1^{d_1} \star \sigma_2^{d_2} \star \dots \star \sigma_i^{d_i} \star \dots \star \sigma_{\mathfrak{s}}^{d_{\mathfrak{s}}}.$$

Notice that the construction of \tilde{u} avec estimate (2.14) carries over to $W^{1, \mathfrak{p}}$ maps by a density argument.

Proof of (2.9) completed. Consider some integer $i \in \{1, \dots, \mathfrak{s}\}$, some number $d \in \mathbb{Z}$ and $u \in W^{1, \mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}, \mathcal{N}^{\mathfrak{p}})$ such that $\langle u \rangle_i = d$. We claim that there exists some constant $C > 0$ which does not depend on u nor on d such that

$$E_{\mathfrak{p}}(u) \geq C|d|. \quad (2.15) \quad \{\text{grouiner}\}$$

Indeed, in view of the results in previous paragraph, we may construct some map $\tilde{u} \in W^{1, \mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}, \mathcal{N}^{\mathfrak{p}})$ such that $E_{\mathfrak{p}}(\tilde{u}) \leq CE_{\mathfrak{p}}(u)$ and $\langle \tilde{u} \rangle_{i, \mathfrak{p}} = d$. Applying (2.13) to \tilde{u} , we are led to $E_{\mathfrak{p}}(\tilde{u}) \geq C|d|$. Combining the previous inequalities we derive the proof of the claim (2.15). Finally, to establish (2.9), it suffices to take the infimum in (2.15) over all maps in the homotopy class. This completes the proof of Lemma 2.2. \square

2.3 Proof of Proposition 4

We deduce from the definition of \mathfrak{p}_c that \mathcal{N} is $(\mathfrak{p}_c - 1)$ -simply connected, so that since (2.1) holds for $\mathfrak{p} = \mathfrak{p}_c$, we are in position to apply both Lemma 2.1 and Lemma 2.2. Combining the lower bound (2.5) with the upper bound (2.9), we derive (33). Then, choosing

$$\mathfrak{v}_d^i = \Phi_d^i \text{ for } d \in \mathbb{Z},$$

we observe that, thanks to (2.2), estimate (34) is satisfied, which completes the proof.

3 Proof of Proposition 2

3.1 Introductory remarks

We define first a few quantities which enter in the proof. For an integer $m \geq 1$, an exponent $p > 1$ and given $u \in T_{\text{race}, q_0}^{m, p}(\mathcal{N})$ we introduce the quantity

$$\mathcal{E}_{m, p}^{\text{xt}}(u) = \inf \{ E_p(U, C_{\text{yld}}^m(3/2)), U \in W_{\text{loc}}^{1, p}(C_{\text{yld}}^m(3/2), \mathcal{N}), U(x, 0) = u(x) \text{ for } x \in \mathbb{R}^{m-1} \}. \quad (3.1) \quad \{\text{credoc}\}$$

It follows from Hölder's inequality that for $p \geq m_c$,

$$\mathcal{I}_m^{\text{xt}}(u) \leq C_m \left(\mathcal{E}_{m, p}^{\text{xt}}(u) \right)^{\frac{\mathfrak{p}_c}{p}}, \quad (3.2) \quad \{\text{wolferine}\}$$

where $\mathcal{I}_m^{xt}(u)$ is defined in (12) and differs from $\mathcal{E}_{m,p}^{xt}$ by the choice of exponents both for the energy and the Sobolev maps, which are respectively \mathfrak{p}_c and $\mathfrak{m}_c = \mathfrak{p}_c + 1$. On the other hand, it follows from the definition (3.1) that we have the inequality

$$\mathcal{E}_{m,p}^{xt}(u) \leq \mathfrak{E}_{m,p}^{xt}(u), \quad (3.3) \quad \{\text{classique}\}$$

the main difference between these two quantities being that the domain of integration of the energy is smaller for the one on the left-hand side. Combining (3.2) with (3.3), we are led to the lower bound for $\mathfrak{E}_{m,p}^{xt}(u)$ given by

$$(\mathcal{I}_m^{xt}(u))^{\frac{p}{\mathfrak{p}_c}} \leq C_{m,p} \mathfrak{E}_{m,p}^{xt}(u), \quad (3.4) \quad \{\text{but}\}$$

where $C_{m,p} > 0$ denotes some constant depending only on m and p . The proof of Proposition 1 relies on a lower bound for $\mathcal{I}_m^{xt}(u)$ for appropriate functions u , which immediately yields a lower bound for $\mathfrak{E}_{m,p}^{xt}(u)$, in view of inequality (3.4). The core of the argument actually deals with the critical dimension $m = \mathfrak{m}_c$ with the choice of the function $u = \mathfrak{v}_d^i$. In several places, in particular when we increase dimensions, we rely on the following lemma:

Lemma 3.1. *Let f given an integrable non-negative function on the cylinder $C_{\text{yld}}(R)$ for some $1 \leq R \leq 2$. We have,*

$$\int_{C_{\text{yld}}(R)} f(x) dx \geq \frac{1}{2} \int_0^R \left(\int_{\Lambda(r)} f(\sigma) d\sigma \right) dr. \quad (3.5) \quad \{\text{lefuneeste}\}$$

Proof. Inequality (3.5) is a consequence of the fact that the cylinder $C_{\text{yld}}(R)$ may be decomposed as $C_{\text{yld}}(R) = \bigcup_{r \in [0, R]} \Lambda(r)$ and of Fubini's theorem (or perhaps more precisely, the coarea formula). \square

3.2 The critical dimension $m = \mathfrak{m}_c$

Lemma 3.2. *We have, for some constant $c_0 > 0$ and any number $d \in \mathbb{Z}$*

$$\mathcal{I}_{\mathfrak{m}_c}^{xt}(\mathfrak{v}_d^i) \geq c_0 |d|.$$

Proof. We first notice that, since the function $v = \mathfrak{v}_d^i$ is Lipschitz, it belongs to the space $T_{\text{race}, q_0}^p(\mathcal{N})$, for any $p \geq 1$. Consider next an arbitrary map $V_d \in W_{\text{loc}}^{1, \mathfrak{m}_c}(\mathcal{D}_{\mathfrak{m}_c}, \mathcal{N})$ such that $V_d(x, 0) = \mathfrak{v}_d^i(x)$ for $x \in \mathbb{R}^{\mathfrak{m}_c-1}$. We are in position to apply Proposition 3 in dimension $m = \mathfrak{m}_c$ to the functions $v = \mathfrak{v}_d^i$ and V_d with $p = \mathfrak{m}_c$ and $s = \mathfrak{p}_c = \mathfrak{m}_c - 1$. It follows, in view of (29) and the lower bound provided by (34), that for every $1 \leq r \leq 2$ we have

$$\int_{\Lambda^{\mathfrak{m}_c-1}(r)} |\nabla V_d|^{\mathfrak{m}_c-1} \geq C_{\mathfrak{m}_c} |d|. \quad (3.6) \quad \{\text{riritou}\}$$

We apply the inequality (3.5) to the function $f = |\nabla V_d|^{\mathfrak{m}_c-1}$. This yields

$$\begin{aligned} \int_{C_{\text{yld}}(3/2)} |\nabla V_d|^{\mathfrak{m}_c-1} dx &\geq \frac{1}{2} \int_1^{3/2} \left(\int_{\Lambda^{\mathfrak{m}_c-1}(r)} |\nabla V_d|^{\mathfrak{m}_c-1} \right) dr \\ &\geq \frac{1}{4} C_{\mathfrak{m}_c} |d|, \end{aligned} \quad (3.7) \quad \{\text{lefuneeste1}\}$$

where, for the inequality on the second line, we have invoked (3.6). On the other hand, we have, in view of the definition of $\mathcal{I}_{m_c}^{\text{xt}}(\mathbf{v}_d^i)$

$$\mathcal{I}_{m_c}^{\text{xt}}(\mathbf{v}_d^i) = \inf \left\{ \int_{C_{\text{yld}}(3/2)} |\nabla V_d|^{m_c-1} dx, V_d \in W_{\text{loc}}^{1,m_c}(\mathcal{D}_{m_c}, \mathcal{N}), \text{s.t. } V_d(x, 0) = \mathbf{v}_d^i(x) \right\},$$

so that the conclusion follows from (3.7). \square

3.3 Adding dimensions

Given an integer $m \in \mathbb{N}^*$, our first task will be to construct⁶ a mapping

$$\mathcal{J}^m : T_{\text{race}, \mathbf{q}_0}^{m,p}(\mathbb{R}^\ell) \rightarrow T_{\text{race}, \mathbf{q}_0}^{m+1,p}(\mathbb{R}^\ell),$$

which, to each map $u : \mathbb{R}^{m-1} \rightarrow \mathbb{R}^\ell$ such that u is constant equal to some value \mathbf{q}_0 outside the unit ball \mathbb{B}^{m-1} , relates a map $\mathcal{J}^m(u) : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$, constant equal to \mathbf{q}_0 outside the unit ball \mathbb{B}^m . This map is obtained by means of a combination of several elementary geometric constructions, in particular a cylindrical rotation. First, we consider the translated map \tilde{u} defined on \mathbb{R}^{m-1} by

$$\tilde{u}(x) = u(x - A^{m-1}) \text{ where } A^{m-1} \text{ denotes the point } A^{m-1} = (2, 0, \dots, 0) \in \mathbb{R}^{m-1},$$

so that \tilde{u} is equal to \mathbf{q}_0 outside the ball $\mathbb{B}_1^{m-1}(A^{m-1}) \subset \mathbb{B}_3^{m-1}(0)$. We then introduce the map $T^m(u)$ defined for $(x_1, x_2, \dots, x_{m-1}, x_m) \in \mathbb{R}^m$ by

$$T^m(u)(x_1, x_2, \dots, x_{m-1}, x_m) = \tilde{u}(\mathbf{r}(x_1, x_2), x_3, \dots, x_{m-1}, x_m),$$

where we have set $\mathbf{r}(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$. It follows by construction that the map $T^m(u)$ possesses cylindrical symmetry around the $(m-2)$ -dimensional hypersurface $x_1 = x_2 = 0$. Moreover, is equal to \mathbf{q}_0 outside the ball \mathbb{B}_3^m of radius 3 and center the origin and actually also on the cylinder $[-\frac{1}{2}, \frac{1}{2}]^2 \times \mathbb{R}^{m-2}$. Since we wish the map $\mathcal{J}^m(u)$ to be constant outside the unit ball \mathbb{B}_1^m , we need normalize the previous map and set

$$\mathcal{J}^m(u)(x) = T^m(u)(3x), \text{ for } x \in \mathbb{R}^m. \quad (3.8)$$

It follows from the above observations that, as desired, the map $\mathcal{J}^m(u)$ equals \mathbf{q}_0 outside \mathbb{B}_1^m and also on the cylinder

$$\mathcal{Q}^m \equiv [-\frac{1}{6}, \frac{1}{6}]^2 \times \mathbb{R}^{m-2}.$$

The reader may easily prove the following:

Lemma 3.3. *The map \mathcal{J}^m is affine and continuous from $T_{\text{race}, \mathbf{q}_0}^{m,p}(\mathbb{R}^\ell)$ to $T_{\text{race}, \mathbf{q}_0}^{m+1,p}(\mathbb{R}^\ell)$. If u is a Lipschitz map in $T_{\text{race}, \mathbf{q}_0}^{m,p}(\mathbb{R}^\ell)$, then $\mathcal{J}^m(u)$ is also Lipschitz with*

$$\|\nabla \mathcal{J}^m(u)\|_{L^\infty(\mathbb{R}^m)} \leq C_m \|\nabla u\|_{L^\infty(\mathbb{R}^{m-1})},$$

where $C_m > 0$ denotes a constant depending only on m .

⁶A similar construction is used in [1].

We next specify somewhat the discussion to \mathcal{N} -valued maps. We have:

Proposition 3.1. *Assume that $m \geq m_c$ and that $u \in \mathbf{T}_{\text{race}, q_0}^{m, m_c}(\mathcal{N})$. Then we have, for some constant $C_m > 0$ depending only on m*

$$\mathcal{I}_{m+1}^{\text{xt}}(\mathfrak{J}^m(u)) \geq C_m \mathcal{I}_m^{\text{xt}}(u). \quad (3.9) \quad \{\text{rotextension}\}$$

Proof. The proof of (3.9) is actually mainly a consequence of Fubini's theorem. In order to see this, we introduce first some notation. For $\theta \in \mathbb{R}$, we consider the vector $\vec{e}_\theta = (\cos \theta, \sin \theta, 0, \dots, 0) = \cos \theta \vec{e}_1 + \sin \theta \vec{e}_2$ of \mathbb{R}^m and set $x_\theta = x \cdot \vec{e}_\theta$, for $x \in \mathbb{R}^m$. We introduce the $(m-1)$ -dimensional hyperplane \mathcal{P}_θ^{m-1} of \mathbb{R}^m defined by

$$\mathcal{P}_\theta^{m-1} \equiv \text{Vect} \{ \vec{e}_\theta, \vec{e}_3, \dots, \vec{e}_m \}$$

and the half-hyperplane $\mathcal{P}_\theta^{m-1,+}$ defined by

$$\mathcal{P}_\theta^{m-1,+} = \{ x \in \mathcal{P}_\theta^{m-1}, x_\theta \equiv x \cdot \vec{e}_\theta \geq 0 \}. \quad (3.10) \quad \{\text{peplum}\}$$

We also consider the ball inside $\mathcal{P}_\theta^{m-1,+}$ centered at the point $\tilde{A}_\theta^{m-1,+} = \frac{2}{3}(\cos \theta, \sin \theta, 0, \dots, 0)$ and of radius $R > 0$ defined by

$$\mathcal{B}_\theta^{m-1,+}(R) = \{ x \in \mathcal{P}_\theta^{m-1,+}, 0 \leq (x_\theta - \frac{2}{3})^2 + x_3^2 + \dots + x_m^2 \leq R^2 \}.$$

As above, if we consider a non-negative function f defined on the domain $\mathbb{R}^m \times [0, 2]$ we have thanks to Fubini's Theorem and for any $0 < R < \frac{2}{3}$

$$\begin{aligned} \int_{\mathbb{R}^m \times [0, 2]} f(x) \, dr &= \int_0^{2\pi} \left(\int_{\mathcal{P}_\theta^{m-1,+} \times [0, 2]} |x_\theta| f(x) dx \right) d\theta \geq \int_0^{2\pi} \left(\int_{\mathcal{B}_\theta^{m-1,+}(R) \times [0, 2]} |x_\theta| f(x) dx \right) d\theta \\ &\geq \left(\frac{2}{3} - R \right) \int_0^{2\pi} \left(\int_{\mathcal{B}_\theta^{m-1,+}(R) \times [0, 2]} f(x) dx \right) d\theta. \end{aligned} \quad (3.11) \quad \{\text{fou}\}$$

Consider next an arbitray map $V \in \mathfrak{W}_{m+1}(\mathfrak{J}^m(u))$, so that V is defined on the $(m+1)$ -dimensional cylinder $C_{\text{yld}}^{m+1}(3/2)$ and satisfies

$$\begin{aligned} V(x_1, x_2, \dots, x_m, 0) &= \mathfrak{J}^m u(x_1, x_2, \dots, x_m) \text{ for } x_1^2 + x_2^2 + x_m^2 \leq 1, x_1 \geq 0 \\ &= \tilde{u}(3\mathfrak{r}(x_1, x_2), 3x_3, \dots, 3x_m). \end{aligned} \quad (3.12)$$

We apply the identity (3.11) to the map

$$f = \mathbf{1}_{C_{\text{yld}}^{m+1}(3/2)} |\nabla V|^{m_c-1} \text{ with radius } R = \frac{1}{2}.$$

This yields

$$\int_{C_{\text{yld}}^{m+1}(3/2)} |\nabla V|^{m_c-1} \geq \frac{1}{6} \int_0^{2\pi} \left(\int_{\mathcal{B}_\theta^{m-1,+}(1/2) \times [0, 2]} |\nabla V|^{m_c-1} dx \right) d\theta. \quad (3.13) \quad \{\text{drone}\}$$

We claim that for any $\theta \in [0, 2\pi]$, we have

$$\int_{\mathcal{B}_\theta^{m-1,+} \times [0, 1/4]} |\nabla V|^{m_c-1} dx \geq C \mathcal{I}_m^{\text{xt}}(u). \quad (3.14) \quad \{\text{claimitude}\}$$

Proof of the claim (3.14). Given an arbitray map v_θ defined on $\mathcal{P}_\theta^{m-1,+} \times [0, 2]$, we define a map $\mathfrak{D}_\theta^m(v_\theta)$ on the set $\mathbb{R}^{m-1} \times [0, 2]$ setting for $(x_1, x_3, \dots, x_{m-1}, x_m, x_{m+1}) \in \mathbb{R}^{m-1} \times [0, 2]$

$$\mathfrak{D}_\theta^m(v_\theta)(x_1, x_3, \dots, x_{m-1}, x_m, x_{m+1}) = v_\theta(x_1 \cos \theta, x_1 \sin \theta, x_3, \dots, x_{m-1}, x_m, x_{m+1}). \quad (3.15) \quad \{\text{duxbellorum}\}$$

It follows from this definition that the energy E_p is conserved in the sense that, for any $p \geq 1$

$$\int_{\mathbb{B}^{m-1}(\tilde{A}_0, 1/2) \times [0, \frac{1}{4}]} |\nabla \mathfrak{D}_\theta^m(v_\theta)|^p = \int_{\mathcal{B}_\theta^{m-1,+} \times [0, 1/4]} |\nabla v_\theta|^p, \text{ where } \tilde{A}_0^{m-1} = (\frac{2}{3}, 0, \dots, 0) \in \mathbb{R}^{m-1}.$$

We apply this construction to the restriction V_θ of the map V to $\mathcal{P}_\theta^{m-1,+} \times [0, 2]$. Since the restriction of the map V on $\mathbb{R}^m \times \{0\}$ is equal to $\mathfrak{I}^m(u)$, we deduce that

$$\mathfrak{D}_\theta^m V_\theta(x', 0) = w(x') = u(3x' - A m_1) \text{ for any } x' = (x_1, x_3, \dots, x_m) \in \mathbb{R}^{m-1}. \quad (3.16) \quad \{\text{vicino}\}$$

We define next the map ζ_θ on $C_{\text{yld}}^m(3/2)$ setting

$$\zeta_\theta(x', s) = \mathfrak{D}_\theta^m V \left(\left(\frac{x'}{3} + \tilde{A}_\theta \right), \frac{s}{3} \right) \text{ for } x' = (x_1, x_3, \dots, x_m) \in \mathbb{R}^{m-1} \text{ and } s \geq 0.$$

It follows from (3.16) that

$$\zeta_\theta(x', 0) = u(x') \text{ for any } x' = (x_1, x_3, \dots, x_m) \in \mathbb{R}^{m-1},$$

so that the map ζ_θ belongs to $\mathfrak{W}_m(u)$ and hence we have the inequality

$$\int_{C_{\text{yld}}^m(3/2)} |\nabla \zeta_\theta|^{m_c-1} \geq \mathcal{I}_m^{\text{xt}}(u). \quad (3.17) \quad \{\text{mya}\}$$

On the other hand, we have

$$\int_{C_{\text{yld}}^m(3/2)} |\nabla \zeta_\theta|^{m_c-1} = \frac{1}{3^{m-m_c+1}} \int_{\mathbb{B}^{m-1}(\tilde{A}_0, 1/2) \times [0, \frac{1}{4}]} |\nabla \mathfrak{D}_\theta^m V|^{m_c-1}. \quad (3.18) \quad \{\text{totoro}\}$$

Combining (3.3), (3.18) and (3.17) we complete the proof of the claim (3.14).

Going back to (3.13), we obtain, combining with (3.14)

$$\int_{C_{\text{yld}}^{m+1}(3/2)} |\nabla V|^{m_c-1} \geq C \mathcal{I}_m^{\text{xt}}(u).$$

Since this lower bound is true for any map V in $\mathfrak{W}_{m+1}(\mathfrak{I}^m(u))$, it holds also for the infimum on that set yielding the desired conclusion (3.9). \square

3.4 Proof of Propostion 2 completed

Recall that we have already defined the map $\mathfrak{U}_{\mathfrak{p}_c}^k$ in the critical dimension $m = m_c$ by formula (35). We define the maps \mathfrak{U}_m^k inductively on the dimension m setting

$$\mathfrak{U}_{m+1}^k = \mathfrak{I}^m(\mathfrak{U}_m^k) \text{ for any } k \in \mathbb{Z}. \quad (3.19) \quad \{\text{pour}\}$$

Combining the result of Lemma 3.3 with the properties of the map \mathfrak{v}_d^i with $d = k^{\mathfrak{p}_c}$ given in Proposition 2 we obtain

$$\|\nabla \mathfrak{U}_m^k\|_{L^\infty(\mathbb{R}^{m-1})} \leq C_m \|\nabla \mathfrak{v}_d^i\|_{L^\infty(\mathbb{R}^{m-1})} \leq C_m k, \quad (3.20) \quad \{\text{ricca}\}$$

whereas Proposition 3.1 yields

$$\mathcal{I}_m^{\text{xt}}(\mathfrak{U}_m^k) \geq C_m \mathcal{I}_m^{\text{xt}}(\mathfrak{v}_d^i). \quad (3.21) \quad \{\text{labs}\}$$

The first inequality in (15) is a direct consequence of (3.20). For the second, we obtain, combining (3.21) with the result of Lemma 3.2

$$\mathcal{I}_m^{\text{xt}}(\mathfrak{U}_m^k) \geq C_m |d| \geq C_m k^{\mathfrak{p}_c},$$

which yields the second inequality in (15) and hence completes the proof.

4 Proof of Proposition 1

In this section we will provide the proofs to Lemma 3 and Lemma 4 and then complete the proof of proposition 1. \{\text{sparrow}\}

4.1 On the trace norm of glued maps

Whereas the energy norm $W^{1,p}$ has a local nature, the *trace norm does not*. This introduces some interaction terms when computing the trace norm of glued maps. In order to estimate this interaction terms, we are led to consider the general situation where we are given a family of points $\{A_i\}_{i \in I}$ in \mathbb{R}^{m-1} , a family of radii $\{r_i\}_{i \in I}$ and maps in the subspace $\mathfrak{X}_{m,p}(\{r_i, A_i\}_{i \in I})$ of $W^{1-1/p,p}(\mathbb{R}^{m-1}, \mathbb{R}^\ell)$ defined by \{\text{radinitude}\}

$$\mathfrak{X}_{m,p}(\{r_i, A_i\}) = \left\{ u \in W^{1-1/p,p}(\mathbb{R}^{m-1}, \mathbb{R}^\ell) \text{ such that } u = 0 \text{ on } \mathbb{R}^{m-1} \setminus \bigcup_{i \in I} \mathbb{B}^{m-1}(r_i, A_i) \right\}.$$

We assume furthermore that the balls $\mathbb{B}^{m-1}(r_i, A_i)$ are well separated, that is we assume

$$|A_i - A_j| \geq 8(r_i + r_j) \text{ for } i \neq j \text{ in } J. \quad (4.1) \quad \{\text{loin}\}$$

Given a map $u \in W^{1-1/p,p}(\mathbb{R}^{m-1}, \mathbb{R}^\ell)$ we also introduce the "localized trace energy"

$$N_{r,a}(u) = \int_{\mathbb{B}^{m-1}(2r,A)} \left(\int_{\mathbb{B}^{m-1}(2r,A)} \frac{|u(x) - u(y)|^p}{|x - y|^{p+m-2}} dx \right) dy \text{ for } a \in \mathbb{R}^{m-1} \text{ and } r > 0,$$

When $u \in \mathfrak{X}_{m,p}(\{r_i, A_i\})$ we will use the notation $N_i(u) = N_{r_i, A_i}(u)$. The next result relates the trace norm to the localized trace energies.

Lemma 4.1. Assume that (4.1) holds and that $u \in \mathfrak{X}_{m,p}(\{r_i, A_i\}) \cap L^\infty(\mathbb{R}^{m-1})$. Then we have, for some constant $C_m > 0$ depending only on m {trouville}

$$|u|_{1-1/p,p}^p \leq \sum_{i \in I} N_i(u) + C_m \|u\|_\infty^p \sum_{i \in I} r_i^{m-p}. \quad (4.2) \quad \text{{localized}}$$

Proof. Set $\Omega = \mathbb{R}^{m-1} \setminus \bigcup_{i \in I} \mathbb{B}^{m-1}(2r_i, A_i)$ and $\Omega_i = \mathbb{R}^{m-1} \setminus \mathbb{B}^{m-1}(2r_i, A_i)$ for $i \in I$. We may decompose, in view of the defining formula (2) the quantity $|u|_{1-1/p,p}^p$ as

$$|u|_{1-1/p,p}^p = \sum_{i \in I} (N_i(u) + K_i(u)) + R(u), \quad (4.3) \quad \text{{decomposons}}$$

where we have set

$$K_i(u) = \int_{\mathbb{B}^{m-1}(2r_i, A_i)} \left(\int_{\Omega_i} \frac{|u(x) - u(y)|^p}{|x - y|^{p+m-2}} dx \right) dy$$

and

$$R(u) = \int_{\Omega} \left(\int_{\mathbb{R}^{m-1}} \frac{|u(x) - u(y)|^p}{|x - y|^{p+m-2}} dx \right) dy.$$

Since $u(x) = 0$ for $x \in \Omega_i$ and $u(y) = 0$ for $y \in \mathbb{B}^{m-1}(2r_i, A_i) \setminus \mathbb{B}^{m-1}(r_i, A_i)$, we deduce that in the integral defining $K_i(u)$, we have

$$\text{if } |u(x) - u(y)| \neq 0, \ x \in \Omega_i \text{ and } y \in \mathbb{B}^{m-1}(2r_i, A_i), \text{ then } |x - y| \geq |x - A_i| - r_i.$$

It follows that, invoking also the definition of Ω_i that

$$\begin{aligned} K_i(u) &\leq C |\mathbb{B}^{m-1}(2r_i, A_i)| \|u\|_\infty^p \int_{2r_i}^\infty \left(\frac{1}{\varrho - r_i} \right)^{p+m-2} \varrho^{m-2} d\varrho \\ &\leq C_m r_i^{m-p} \|u\|_\infty^p. \end{aligned} \quad (4.4) \quad \text{{grassouillet}}$$

We argue somewhat similarly for $R(u)$. Since $u(y) = 0$ for $y \in \Omega$ it follows that

$$\begin{aligned} R(u) &= \int_{\Omega} \left(\int_{\mathbb{R}^{m-1}} \frac{|u(x)|^p}{|x - y|^{p+m-2}} dx \right) dy = \sum_{i \in I} \int_{\Omega} \left(\int_{\mathbb{B}^{m-1}(r_i, A_i)} \frac{|u(x)|^p}{|x - y|^{p+m-2}} dx \right) dy \\ &\leq \|u\|_\infty^p \sum_{i \in I} \int_{\Omega} \left(\int_{\mathbb{B}^{m-1}(r_i, A_i)} \frac{dx}{|x - y|^{p+m-2}} \right) dy \\ &\leq \|u\|_\infty^p \sum_{i \in I} \int_{\mathbb{B}^{m-1}(r_i, A_i)} \left(\int_{\Omega} \frac{dy}{|x - y|^{p+m-2}} \right) dx. \end{aligned} \quad (4.5) \quad \text{{grasdoble}}$$

Since $\text{dist}(\mathbb{B}^{m-1}(r_i, A_i), \Omega) \geq r_i$, we deduce, that for $x \in \mathbb{B}^{m-1}(r_i, A_i)$ we have

$$\int_{\Omega} \frac{dy}{|x - y|^{p+m-2}} \leq C_m \int_{r_i}^{+\infty} \frac{1}{\varrho^{p+m-2}} \varrho^{m-2} d\varrho \leq C_m r_i^{-p+1}.$$

Going back to (4.5) we are hence led to

$$R(u) \leq C_m \|u\|_\infty^p \sum_{i \in I} |\mathbb{B}^{m-1}(r_i, A_i)| r_i^{-p+1} \leq C_m r_i^{m-p}. \quad (4.6) \quad \text{{dobile}}$$

Combining (4.4) and (4.6) with (4.3) we obtain the desired conclusion (4.2). \square

4.2 Proof of Lemma 3

{thracity}

We apply the result of Lemma 4.1 to the case $I = \mathbb{N}$, $r_i = \mathfrak{r}_i$ and $A_i = \mathfrak{M}_i$ for $i \in \mathbb{N}$, so that the map $\mathbf{u}_{\text{obst}} - \mathbf{q}_0$ constructed in (40) with respect to the given sequences $(\mathfrak{r}_i)_{i \in \mathbb{N}}$ and $(\mathfrak{M}_i)_{i \in \mathbb{N}}$ belongs to $\mathfrak{X}_{m,p}(\{\mathfrak{r}_i, \mathfrak{M}_i\}_{i \in I})$. It also belongs to $L^\infty(\mathbb{R}^{m-1})$ since \mathbf{u}_{obst} is \mathcal{N} valued. It follows from the second assumption in (41) that (4.1) is satisfied. We are hence in position to apply inequality (4.2) to \mathbf{u}_{obst} . It yields

$$\|\mathbf{u}_{\text{obst}} - \mathbf{q}_0\|_{1-1/p,p}^p \leq \sum_{i \in \mathbb{N}} N_i(\mathbf{u}_{\text{obst}} - \mathbf{q}_0) + C_m L^p \sum_{i \in \mathbb{N}} \mathfrak{r}_i^{m-p}. \quad (4.7) \quad \{\text{loc}\}$$

In view of the definition (40) and the scaling law (38), we have

$$N_i(\mathbf{u}_{\text{obst}}) \leq \mathfrak{r}_i^{m-p} |\mathfrak{U}_m^k - \mathbf{q}_0|_{1-\frac{1}{p},p}^p \leq C_m \mathfrak{r}_i^{m-p} k_i^{p-1},$$

so that going back to (4.7) we obtain

$$\|\mathbf{u}_{\text{obst}} - \mathbf{q}_0\|_{1-1/p,p}^p \leq C_m \sum_{i \in \mathbb{N}} \mathfrak{r}_i^{m-p} (k_i^{p-1} + 1). \quad (4.8) \quad \{\text{glouglou}\}$$

Since the right hand side of this inequality is finite in view of assumption (41), the conclusion follows. \square

4.3 Proof of Lemma 4

We may assume that the set

$$\mathcal{Z}_{m,p} = \{U \in W_{\text{loc}}^{1,p}(\mathcal{D}_m, \mathcal{N}), U(x, 0) = \mathbf{u}_{\text{obst}}(x) \text{ for } x \in \mathbb{R}^{m-1}\}$$

is not empty since otherwise $\mathfrak{E}_{m,p}^{\text{xt}}(u) = +\infty$ and the proof is complete in that case. Let $U \in \mathcal{Z}_{m,p}$. As a consequence of the definition (40) and the scaling law (38)

$$\begin{aligned} E_p \left(U, C_{\text{yld}} \left(\frac{3}{2} \mathfrak{r}_i \right) + \mathfrak{M}_i \right) &\geq \mathfrak{r}_i^{m-p} \mathcal{E}_{m,p}^{\text{xt}}(\mathfrak{U}_m^{k_i}) \geq C_m \mathfrak{r}_i^{m-p} \left(\mathcal{I}_m^{\text{xt}}(\mathfrak{U}_m^{k_i}) \right)^{\frac{p}{p_c}} \\ &\geq C_m \mathfrak{r}_i^{m-p} k_i^p. \end{aligned} \quad (4.9)$$

Since the collection of sets $(C_{\text{yld}}(\frac{3}{2}\mathfrak{r}_i) + \mathfrak{M}_i)_{i \in \mathbb{N}}$ represents a collection of disjoint sets, we may sum up the previous inequalities, which leads to the inequality

$$E_p(U, \mathcal{D}_m) \geq \sum_{i \in \mathbb{N}} C_m \mathfrak{r}_i^{m-p} k_i^p.$$

Taking the infimum over all maps in $\mathcal{Z}_{m,p}$ we obtain the desired conclusion. \square

4.4 Proof of Proposition 1 completed

We claim that there exists a sequence of real positive numbers $(\mathfrak{r}_i)_{i \in \mathbb{N}}$ and a sequence of integers $(k_i)_{i \in \mathbb{N}}$ such that both (41) and (42) are satisfied. There is a large variety of possible choices for such sequences, here we propose one of them. Setting for instance

$$\mathfrak{r}_i = \left(\frac{1}{i+1} \right)^{\frac{p+1}{m-p}} \text{ and } k_i = i+1. \quad (4.10) \quad \{\text{seteq}\}$$

we verify that this choice satisfies assumptions (41) and (42). With this choice of sequences, it follows from Lemma 3 that \mathbf{u}_{obst} belongs to $W^{1-1/p,p}(\mathbb{R}^{m-1}, \mathcal{N})$, whereas Lemma 4 shows that

$$\mathfrak{E}_{m,p}^{\text{xt}}(\mathbf{u}_{\text{obst}}) = +\infty$$

and hence has no finite energy extension, completing the proof of Proposition 1. \square

5 Proof of Theorem 1

{black}

We choose an arbitrary point of A_0 on \mathcal{M} . Given $\varrho > 0$, we consider the geodesic ball on \mathcal{N} centered at A_0 and of radius ϱ given by

$$B_{\text{geod}}(\varrho, A_0) = \{x \in \mathcal{M} \text{ such that } \text{dist}_{\text{geod}}(x, A_0) < \varrho\},$$

where $\text{dist}_{\text{geod}}$ stands for the geodesic distance on \mathcal{N} . By standard results, there exists some $\varrho_0 > 0$ and a diffeomorphism $\Phi : \mathbb{B}_m^+(2) \rightarrow B_{\text{geod}}(\varrho_0, A_0)$ such that

$$\Phi(\mathbb{B}^{m-1}(2) \times \{0\}) = B_{\text{geod}}(\varrho_0, A_0) \cap \partial\mathcal{M},$$

with

$$\mathbb{B}_m^+(2) = \{x = (x', x_m) \in \mathbb{B}^m(2), \text{ with } x' \in \mathbb{R}^{m-1}, x' \geq 0\}.$$

Assume next that $\mathfrak{p}_c + 1 \leq p < m$. We define a map $\mathbf{w}_{\text{obst}} \in W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$ setting

$$\begin{cases} \mathbf{w}_{\text{obst}}(x) = \mathbf{u}_{\text{obst}}(\Phi^{-1}(x)) \in x \in B_{\text{geod}}(\varrho, A_0) \cap \partial\mathcal{M}, \\ \mathbf{w}_{\text{obst}}(x) = \mathbf{q}_0 \text{ otherwise.} \end{cases} \quad (5.1)$$

We claim that there exists no map $W \in W^{1,p}(\mathcal{M}, \mathcal{N})$ such that $W(x) = \mathbf{w}_{\text{obst}}(x)$ on $\partial\mathcal{M}$. Indeed, assume by contradiction that such a map W does exist. Let \tilde{W} be the restrict of the map W to the set $B_{\text{geod}}(\varrho_0, A_0)$. Then the map $U = \Phi^{-1} \circ \tilde{W}$ would belong to $W^{1,p}(\mathbb{B}_m^+(2), \mathcal{N})$ with

$$\tilde{W}(x) = \mathbf{u}_{\text{obst}}(x) \text{ for } x \in \mathbb{B}^{m-1}(2) \times \{0\}.$$

This however contradicts the properties of \mathbf{u}_{obst} as stated in Proposition 1 and hence shows that for $\mathfrak{p}_c + 1 \leq p < m$, the extension property does not hold. For the existence part, that is when $1 < p < \mathfrak{p}_c + 1$ we invoke the result in [10], to assert that the existence property holds, so that the proof of Theorem 1 is complete. \square

6 The case \mathcal{N} is not simply connected

{pearl}

In this section, we provide the proofs of Theorem 2 and Theorem 3.

6.1 Proof of Theorem 2

We assume that $2 \leq p < m$ and prove that, if the assumptions of Theorem 2 are satisfied, then, in that case, the extension property $\text{Ext}_p(\mathcal{M}, \mathcal{N})$ does not hold. This is indeed a consequence of Proposition 2 in [2], which we briefly recall: It asserts that, given $0 < s < 1$ and $p \geq 1$ such that $1 \leq sp < m - 1$ and assuming that $\pi_1(\mathcal{N})$ is infinite, then there exists a

map $u \in W^{s,p}(\partial\mathcal{M}, \mathcal{N})$ such that u can not be written as $u = \pi \circ \varphi$, with $\varphi \in W^{s,p}(\partial\mathcal{M}, \mathcal{N}_{\text{cov}})$. We apply this result to the specific case which is of interest for us, namely the case $s = 1 - 1/p$, so that $m - 1 > sp = p - 1 \geq 1$. Proposition 2 in [2] hence shows that $\mathcal{L}_p^{\text{ift}}(\partial\mathcal{M}, \mathcal{N})$ does not hold, and therefore nor does the extension property $\text{Ext}_p(\mathcal{M}, \mathcal{N})$, in view of Lemma 1. The proof is hence complete. \square

6.2 Proof of Theorem 3

{fuseaux}

We proceed distinguishing four cases.

Case 1: $\tilde{\mathfrak{p}}_c + 1 \leq p < m$. It follows from Theorem 1 that $\text{Ext}_p(\mathcal{M}, \mathcal{N}_{\text{cov}})$ does not hold, hence there exists some map $\varphi \in W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N}_{\text{cov}})$ which cannot be extended as $W^{1,p}(\mathcal{M}, \mathcal{N}_{\text{cov}})$ map to the whole of \mathcal{M} . Next we set

$$u = \Pi \circ \varphi, \text{ so that } u \in W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N}).$$

We claim that u can not be extended as a $W^{1,p}(\mathcal{M}, \mathcal{N})$ map to the whole of \mathcal{M} . To prove the claim, we assume by contradiction that there exists some map $U \in W^{1,p}(\mathcal{M}, \mathcal{N})$ such that $U(\cdot) = u(\cdot)$ on the boundary $\partial\mathcal{M}$. Since $p \geq 2$, it follows from Theorem 1 in [2] that there exists some map $\Phi \in W^{1,p}(\mathcal{M}, \mathcal{N}_{\text{cov}})$ such that $U = \pi \circ \Phi$. Restricting this relation to the boundary, we are led to $\Phi(\cdot) = \varphi(\cdot)$ on $\partial\mathcal{M}$, contradicting the fact that φ cannot be extended and hence proving the claim. It follows that $\text{Ext}_p(\mathcal{M}, \mathcal{N})$ does not hold, establishing the first assertion in part i) Theorem 3.

Case 2 : $1 \leq p < \tilde{\mathfrak{p}}_c + 1 \leq m$ and property $\mathcal{L}_p^{\text{ift}}(\partial\mathcal{M}, \mathcal{N})$ holds. We will show that in that case, given any map $u \in W^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$ there exists a map $U \in W^{1,p}(\mathcal{M}, \mathcal{N})$ such that $U(\cdot) = u(\cdot)$ on the boundary $\partial\mathcal{M}$. Since we assume that $\mathcal{L}_p^{\text{ift}}(\partial\mathcal{M}, \mathcal{N})$ holds, there exists some map $\varphi \in W^{1,p}(\partial\mathcal{M}, \mathcal{N}_{\text{cov}})$ such that $u = \Pi \circ \varphi$. Applying Theorem 1 to the target \mathcal{N}_{cov} , we see that property $\text{Ext}_p(\mathcal{M}, \mathcal{N}_{\text{cov}})$ holds, so that there exist a map $\Phi \in W^{1,p}(\mathcal{M}, \mathcal{N}_{\text{cov}})$ such that $\Phi(\cdot) = \varphi(\cdot)$ on $\partial\mathcal{M}$. Setting $U = \pi \circ \Phi$, we obtain the desired map U . This proves that $\text{Ext}_p(\mathcal{M}, \mathcal{N})$ holds in the case considered. As a special case, we obtain the part iii) of Theorem 3.

Case 3: $1 \leq p < 2$. In this special case, it follows from Theorem 3 case ii) of [2] applied with $s = 1 - 1/p$, so that $sp = p - 1 < 1$, that $\mathcal{L}_p^{\text{ift}}(\partial\mathcal{M}, \mathcal{N})$ holds. Hence the assumptions of Case 2 are satisfied, so that we obtain that $\text{Ext}_p(\mathcal{M}, \mathcal{N})$ holds in the case considered. This yields the proof to part ii) of Theorem 3.

Case 4: $2 \leq 3 \leq p < m$. In this case, since $\pi_{[p]-1}(\mathcal{N}) = \pi_1(\mathcal{N}) \neq \{0\}$, we obtain by [3] (using the method of [10]) topological obstructions to the extension problem, yielding hence the proof to the second statement in Theorem 3, part i).

The three parts of Theorem 3 are hence proved, so that the proof is complete. \square

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